A Generalization of the Sherrington-Kirkpatrick Model

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Abstract

The Sherrington-Kirkpatrick model is one of the most important models in spin glass theory. We consider a generalization with multivariate interactions. Several common methods are not conclusive. We rigorously treat the model at high enough temperature based on [15]. For infinite alphabets, Gaussian Random Fields for the interaction are considered. Furthermore we give conjectures on the Parisi-type formula.

Zusammenfassung

Das Sherrington-Kirkpatrick Modell ist eines der wichtigsten Modelle auf dem Gebiet der Spin Gläser. Wir betrachten eine Verallgemeinerung mit multivariaten Interaktionen. Hier sind gängige Beweisemethoden nicht mehr schlüssig. Rigoros behandeln wir den Fall genug hoher Temperatur basierend auf [15]. Für unendliche Spinmengen werden Gausssche Zufallsfelder für die Interaktionen betrachtet. Darüberhinaus geben wir Vermutungen über die Parisi-Formel an.

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Part I.

The Multivariate SK Model – an Introduction

1. The Multivariate Sherrington-Kirkpatrick Model

The Sherrington-Kirkpatrick (SK) model was introduced in the 70ies as a 'Solvable Model of Spin Glasses' [13]. Most definitely, physicists have been able to gain much insight into a rich structure this model exhibits. But some of this solution still is withstanding all mathematical attacks. Only in recent years several paths of rigorous proof penetrated the world of truth this model harbors as a challenge for mathematicians.

The purpose of this thesis lies in sharpening the mathematical methods developed in recent years to attack this model. We do this by throwing some of those techniques at a more general version of the SK model – maybe the 'most general SK model' on a finite alphabet of spins. We even will consider infinite alphabets at some point.

Why could this be of any help? Being one of the main mean field models of statistical mechanics since its introduction by Sherrington and Kirkpatrick, the first improvement over the so called 'Replica Symmetric Solution' were the predictions by Parisi – namely his free energy formula and ultrametricity found by the notorious method called 'Replica Symmetry Breaking'. Mathematically the formula has been proved only quite recently in works by Guerra and Talagrand. The hierarchical structure is still under much debate. Even physicists are not yet completely satisfied by their work and resort to numerical simulations to gather new evidence. Parisi and his collaborators were even forced to develop methods of supercomputing to be able to get satisfying results, cf. [10].

The mathematics of the SK model was mostly improved by thoroughly understanding the high temperature case with non-vanishing external field. This is because otherwise symmetry conceals effects that dominate at low temperatures. Therefore even better understanding of the high temperature regime may empower mathematicians to gain proofs of conjectures physicists have taken for granted for decades.

Therefore, we examine a generalization of the SK model on finite state space. The SK model – being a mean field version of the Edwards-Anderson model – has Gaussian interaction. The most natural and comprehensive extension onto any finite spin state space Σ then is a matrix of joint centered Gaussian variables $\big(g(s,t)\big)_{s,t\in\Sigma}$. This is what we will consider here.

And in fact, this can give much insight into the structures of the standard SK model. For instance reminiscents of the Parisi formula appear in this case which vanish in the standard case. On the other hand, there are even models that do not exhibit Replica Symmetry Breaking, that is: no phase transition! From the viewpoint of methodology we have to point out that out of several ways to prove the convergence of the overlap that work for the SK model, only the proof introduced in [15] turned out to be powerful enough for the task – on account of the more complex structure of overlap in this case.

In this section, we will first present the model using a linguistic motivation and then state the details of the definition as well as some notation and preliminaries. After a short heuristical explanation of some of the issues of the definitions, we present the results of this thesis followed by some conjectures towards lower temperature.

In the next chapter we will give several examples for the multivariate SK model. We finish this part by giving some exemplary calculations which will be used on a regular basis later on.

1.1. Motivation

To motivate the multivariate SK model, consider the following model of neologisms in a community of N speakers. Say there is a meaning having two or more words for it. Most of the N speakers are using the traditional word A, but there are some who use more modern words B, C etc. for it. We denote the set of words under consideration $\Sigma := \{A, B, C, \ldots\}$ and assume this set to be finite. We assume that each speaker has to make a choice what word he will use and consider the set of choices

$$\Sigma^N := \{\, \boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \mid \sigma_1, \ldots, \sigma_N \in \Sigma \,\}.$$

That is, a choice $\boldsymbol{\sigma}$ gives for every member i of the community his choice σ_i. Since language is mainly used as a two way channel, social factors play an important role influencing the probability of observing a given choice. We give a model for those probabilities.

But first, we examine a more concrete example on $\Sigma = \{A, B\}$.

Example 1.1. *Let $\Sigma = \{A, B\}$. What might be these social factors? If i is speaking with j*

- *Depending on their relationship i might want to be understood well or badly by its interlocutor j. Call this factor $a_{ij} \in \mathbb{R}$. Then, $a_{ij} > 0$ means that i wants to be understood well by j, and $a_{ij} < 0$ the opposite. The absolute value gives the intensity of the intent.*
- *If i wants to appear in front of the interlocutor j as modern (traditional), using B will favor (impede) its goal. We will call this $b_{ij} \in \mathbb{R}$. If $b_{ij} > 0$ ($b_{ij} < 0$) then i wants to use the modern (traditional) word with j.*

Now, we measure the achievement of i's goals with the assessment function:

$$f_{ij}(s,t) = a_{ij} \cdot (1_{s=t} - 1_{s \neq t}) + b_{ij} \cdot 1_{s=B}, \qquad s, t \in \{A, B\}$$

To explain this: assume i uses A and j uses B, then we get $f_{ij}(A, B) = -a_{ij}$. Or if both i and j use B this is $f_{ij}(B, B) = a_{ij} + b_{ij}$. Hence, using the same word as j changes the assessment by $2a_{ij}$ and using B changes it by b_{ij}.

Knowing all those social intents $(a_{ij})_{ij}$ and $(b_{ij})_{ij}$ we model the probability of a choice $\boldsymbol{\sigma} \in \Sigma^N$ using the following total assessment:

$$H(\boldsymbol{\sigma}) = \sum_{i,j=1}^{N} f_{ij}(\sigma_i, \sigma_j), \qquad \boldsymbol{\sigma} \in \{A, B\}^N$$

We could understand these assessments also in terms of energy: If I can fulfill my goal by a simple change of use of my vocabulary, I can use more of my energy for other goals. This will make this choice more likely than others if all others keep their choices. The question then is whether the total energy benefits as well.

Of course it might and will happen that i cannot fulfill all of its goals. If I want to impress my friends using one of the new words, but accommodate my grandmother by using the old word, I will be frustrated by either choice. But still, there will be some choices which will be less frustrating than others and therefore on the long run might be more probable of being observed.

We now generalize this example. Assume for all i and j we have a matrix $\big(g_{ij}(s,t)\big)_{s,t}$ which gives the energy needed when i uses s and j uses t. Then the total energy or assessment is defined as:

$$H(\boldsymbol{\sigma}) = \sum_{\substack{i,j=1 \\ i\neq j}}^{N} g_{ij}(\sigma_i, \sigma_j), \qquad \boldsymbol{\sigma} \in \Sigma^N$$

First let $p(\cdot)$ be the a priori distribution of the words in the general population. Then, as there are exponentially many possible choices, we have to assign to a better choice an exponentially better weight. This leads to the following definition of a probability for every choice $\boldsymbol{\sigma} \in \Sigma^N$:

$$P(\boldsymbol{\sigma}) = \frac{1}{Z} \cdot e^{H(\boldsymbol{\sigma})} p(\boldsymbol{\sigma}), \quad Z := \sum_{\boldsymbol{\sigma}' \in \Sigma^N} e^{H(\boldsymbol{\sigma}')} p(\boldsymbol{\sigma})$$

The normalizing factor Z shows how the probabilities split up the sure event based on the social structure parameters. Therefore, it is usually called the **partition function**.

Observe that this definition epitomizes the eternal competition between quality and quantity, or between energy and entropy: even though there may be some types of choices with excellent energy, there still will be some classes of choices that overcompensate their mediocre energy by an overwhelming mass of possible choices and become predominant. Physicists approach this by looking at the so called **free energy** per spin that is:

$$f := \frac{1}{N} \ln Z = \frac{1}{N} \ln \sum_{\boldsymbol{\sigma}} e^{H(\boldsymbol{\sigma})} p(\boldsymbol{\sigma})$$

Even without any knowledge of statistical mechanics, this is easily seen to be an important quantity. Indeed, as the partition function consists of an average of exponential weights, its logarithm divided by the number of spins stands the chance of actually converging to some finite value as $N \to \infty$.

How is this linked to the battle of energy and entropy? Naturally more interesting than the probability of a single choice, are the probabilities of observable events. E.g. for the event that the system is at energy x, the logarithm of the probability is:

$$\ln P_x = \ln\left[\sum_{\boldsymbol{\sigma}\in\Sigma^N:\ H(\boldsymbol{\sigma})=x} \frac{e^{H(\boldsymbol{\sigma})}}{Z}\right] = x + \ln|\{\,\boldsymbol{\sigma}\in\Sigma^N:\ H(\boldsymbol{\sigma}) = x\,\}| - f$$

and this turns out to correspond to the physicists' definition of the free energy of the event of being at energy x. Furthermore, one can derive several important facts about the system if one knows the partition function or the free energy.

Now, remember that the probability still depends on the social structure. Hence, to complete our model, we have to give a means of producing such structures. This will be done by random variables.

What should those random variables look like? We might be tempted to state: 'Each social factor is the sum of many small decisions'. Therefore, here and in the remainder of this work, we assume that the $\big(g_{ij}(\cdot,\cdot)\big)_{ij}$ are sampled as independent centered Gaussian fields with given covariances:

$$\Gamma(s,t,s',t') := \mathbb{E} g_{ij}(s,t) g_{ij}(s',t'),$$

where by $\mathbb{E}$ we denote as in this whole work the expectation w.r.t. Gaussian variables. Obviously, the joint energy $g_{ij}(s,t)+g_{ji}(t,s)$ is again some centered Gaussian variable. Therefore, we can without loss of generality take this sum and look just at one field per pair. In order to have e^H of the same order as the size of configurations, we will use the following Hamiltonian:

$$H_N(\boldsymbol{\sigma}) = \frac{\beta}{\sqrt{N}} \sum_{\substack{i,j=1\\ i<j}}^{N} g_{ij}(\sigma_i,\sigma_j).$$

Then, the probabilities $P(\cdot)$, Z and the free energy become random variables.

Remark that we now have two sources of randomness: the social structure is random and gives rise to the randomness in the choices. We will denote expectation with respect to social structure and random choice configurations by $\mathbb{E}$ and $\langle\,\cdot\,\rangle$, respectively.

To finish this section we calculate Γ based on the example above.

Example 1.2. *Assume that a_{ij} and b_{ij} are independent centered standard*

Gaussian variables. Then:

$$\begin{aligned} C(s,t,s',t') &:= \mathbb{E} f_{ij}(s,t) f_{ij}(s',t') \\ &= (1_{s=t} - 1_{s\neq t}) \cdot (1_{s'=t'} - 1_{s'\neq t'}) + 1_{s\neq A} \cdot 1_{s'\neq A} \end{aligned}$$

This can be written by a matrix:

$$C = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix}$$

Now for the reverse interaction of j with i we have to permute those entries. This is the matrix $\bar{C}(s,t,s',t') := \mathbb{E} f_{ji}(s,t) f_{ji}(s',t')$, or

$$\bar{C} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$

Observe that the original asymmetric form in $H(\boldsymbol{\sigma})$ splits into independent summands of the form:

$$g_{ij}(s,t) = f_{ij}(s,t) + f_{ji}(t,s)$$

Therefore, $H(\boldsymbol{\sigma}) = \sum_{1\leq i<j\leq N} g_{ij}(\sigma_i, \sigma_j)$ and the g_{ij} are just i.i.d. centered Gaussian fields with covariance matrix:

$$\Gamma = C + \bar{C} = \begin{pmatrix} 2 & -2 & -2 & 2 \\ -2 & 3 & 2 & -1 \\ -2 & 2 & 3 & -1 \\ 2 & -1 & -1 & 4 \end{pmatrix}.$$

Then the free energy can be calculated by the formulas in the next sections.

1.2. The Model

In this work we treat a class of models that embraces both this model, as well as its well studied cousin – the SK model.

As stated before, the most natural and comprehensive extension of the SK model interaction onto any finite spin state space Σ is a matrix of joined centered Gaussian variables $\big(g(s,t)\big)_{s,t\in\Sigma}$. This is what we will consider here.

To give the distribution of such a field is the same as to give its positive semidefinite symmetric covariance matrix

$$\Gamma(s,t,s',t') := \mathbb{E}g(s,t)\cdot g(s',t'), \qquad \gamma(s,t) := \Gamma(s,t,s,t),$$

where we assume $\Gamma(s,t,s',t') = \Gamma(t,s,t',s')$, that is the distribution of the Gaussian field has to be invariant under taking its transpose:

$$\big(g(s,t)\big)_{s,t} \overset{\mathcal{L}}{=} \big(g(t,s)\big)_{s,t}. \tag{1.1}$$

The other piece of data we will assume as given, is the a priori distribution $\big(p(s)\big)_{s\in\Sigma}$ where $\sum_{s\in\Sigma} p(s) = 1$.
Given the matrix Γ, the probability distribution $p(\cdot)$, the inverse temperature $\beta \geq 0$, and some $N \in \mathbb{N}$, we define the model as follows. First let $\big(g_{ij}(\cdot,\cdot)\big)_{i<j\leq N}$ be i.i.d. copies of a Gaussian field with covariance matrix Γ, i.e.:

$$\mathbb{E}g_{ij}(s,t)g_{i'j'}(s',t') = \Gamma(s,t,s',t')\cdot 1_{\{i,j\}=\{i',j'\}}$$

We call $\big(g_{ij}(\cdot,\cdot)\big)_{i<j\leq N}$ the **disorder**. Then we define the **Hamiltonian**:

$$H_N(\boldsymbol{\sigma}) := \frac{\beta}{\sqrt{N}} \sum_{\substack{i,j=1\\ i<j}}^{N} g_{ij}(\sigma_i,\sigma_j), \qquad \boldsymbol{\sigma} \in \Sigma^N \tag{1.2}$$

This induces the following **Gibbs probability distribution** on the **configurations $\boldsymbol{\sigma} \in \Sigma^N$**:

$$P(\boldsymbol{\sigma}) := \frac{e^{H_N(\boldsymbol{\sigma})}}{Z_N}\cdot\prod_{i=1}^{N} p(\sigma_i), \qquad Z_N := \sum_{\boldsymbol{\sigma}'\in\Sigma^N} e^{H_N(\boldsymbol{\sigma}')}\cdot\prod_{i=1}^{N} p(\sigma'_i) \tag{1.3}$$

where Z_N is the **partition function**. Actually, most of our proofs in Part II will have to be done for a slightly more general version that will be defined in (4.5).

1.2.1. Notations

We have three stages of expectation in our framework that will be distinguished. They are defined in terms of a common tool in the theory of Spin

Glasses: fix a single disorder $\big(g_{ij}(\cdot,\cdot)\big)_{i<j\le N}$ and then take independent copies of the corresponding Gibbs measure, that is $P^{\otimes n}(\cdot)$. Those random variables $\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n$ are then called **replicas** to emphasize that they belong to i.i.d. copies of a Gibbs probability given all by the same disorder.

Notation 1.3. *Let $f\colon (\Sigma^N)^n \to \mathbb{R}$ be a function dependent on n replicas.*

(a) *The expectation w.r.t. the a-priori probability is:*

$$\mathrm{Tr}_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n\in\Sigma^N} f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n) := \sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n\in\Sigma^N} f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)\prod_{\ell=1}^{n} p^{\otimes N}(\boldsymbol{\sigma}^\ell)$$

that is the p-average or trace. Using this, we can rewrite the partition function in (1.3) *as:*

$$Z_N := \mathrm{Tr}_{\boldsymbol{\sigma}'\in\Sigma^N}\, e^{H_N(\boldsymbol{\sigma}')} \tag{1.4}$$

(b) *The* ***quenched expectation*** *is integration w.r.t. to the Gibbs probability* (1.3)*:*

$$\langle\, f\,\rangle := \mathrm{Tr}_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n\in\Sigma^N} f(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)\cdot\frac{e^{\sum_{\ell=1}^n H_N(\boldsymbol{\sigma}^\ell)}}{Z_N^n}$$

(c) *The* ***averaged expectation*** *is the joined expectation of $\mathbb{E}\cdot$ and $\langle\,\cdot\,\rangle$:*

$$\nu(f) := \mathbb{E}\,\langle\, f\,\rangle\,.$$

Remark that we integrate in this expression n replicas but only one disorder.

We will often consider the $\boldsymbol{\sigma}^\ell$ to be the natural projections. Hence, $\langle\, \sigma_N^1\cdot\sigma_N^2\,\rangle$ and $\nu(\sigma_N^1\cdot\sigma_N^2)$ are understood in the obvious way.

This work uses heavily a couple of non-standard notations:

Notation 1.4. (a) *Often we will look at a kind of 'transpose' $\Gamma^\star$ of Γ which is defined as follows:*

$$\Gamma^\star(s,s';t,t') := \Gamma(s,t,s',t'),\qquad s,t,s',t'\in\Sigma$$

Remark that $\Gamma^\star$ is usually symmetric, as this is equivalent to (1.1)*. In general though, $\Gamma^\star$ is not positive semidefinite.*

(b) *We need an abuse of notation that will lighten our calculations throughout these notes: Given a function* $f\colon \Sigma^{n+m} \to \mathbb{R}$ *and a measure* η *on* Σ^n *we write:*

$$\begin{aligned} f(\eta, x_1, \ldots, x_m) &:= \int_{\Sigma^n} f(s_1, \ldots, s_n, x_1, \ldots, x_m)\, d\eta(s_1, \ldots, s_n) \\ &= \sum_{s_1, \ldots, s_n \in \Sigma} f(s_1, \ldots, s_n, x_1, \ldots, x_m) \cdot \eta(s_1, \ldots, s_n) \end{aligned}$$

This is done repeatedly and independently for every occurrence of a measure η' *in a given expression. If e.g. we have some distribution* κ *on* Σ^2, *we have:*

$$\begin{aligned} \Gamma^\star(\kappa; \kappa) &= \sum_{s,s',t,t'} \Gamma^\star(s, s'; t, t') \cdot \kappa(s, s')\kappa(t, t') \\ &= \sum_{s,t,s,t'} \Gamma(s, t, s', t') \cdot \kappa(s, s')\kappa(t, t') \end{aligned}$$

(c) *Given a symmetric matrix* $(A_{t,t'})_{t,t' \in \Sigma}$, *we define the matrix* Γ_A *by:*

$$\Gamma_A(s, s') := \Gamma^\star(s, s'; A) = \sum_{t,t'} \Gamma(s, t, s', t') A_{t,t'}$$

(d) *Given a symmetric matrix* $(A_{t,t'})_{t,t' \in \Sigma}$ *with non-negative entries, s.t.* $\sum_{t,t'} A_{t,t'} = 1$, *we say*

$$0 \overset{\Gamma}{\prec} A$$

if and only if Γ_A *is positive semidefinite.*

Remark that $0 \overset{\Gamma}{\prec} A$ follows if A is positive semidefinite.

Easily seen, the notation in *(b)* is just regular vector-matrix multiplication. But because the 'vectors' here often will have the form of matrices, this notation is much clearer. We will use this notation often with the following measures on Σ or Σ^2:

Notation 1.5. (a) *For every spin i, define as an abbreviation:*

$$\delta_i(s) := \delta_{\sigma_i}(s) = 1_{\sigma_i = s}, \qquad s \in \Sigma.$$

In the case of more then one replica, we indicate the dependence in a superscript:

$$\delta_i^\ell(s) := \delta_{\sigma_i^\ell}(s)$$

(b) *The marginal distribution of a spin under the Gibbs-measure is an important quantity, hence for $i \leq N$ and $s \in \Sigma$ we define:*

$$\mu_i(s) := \langle\ \delta_i(s)\ \rangle\,.$$

(c) *We define the empirical distribution as the following random variable:*

$$L_N(s) := \tfrac{1}{N}\sum_{i=1}^{N}\delta_i(s), \qquad s \in \Sigma,$$

that is the average occurrence of s in the configuration $\boldsymbol{\sigma}$. It depends on both sources of randomness, i.e. the Gibbs probability and the disorder. We will use this also for the empirical coincidence of s and s' in $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$:

$$L_N(s,s') := \tfrac{1}{N}\sum_{i=1}^{N}\delta_i(s)\cdot\delta_i'(s'), \qquad s,s' \in \Sigma.$$

Again, to specify which random configurations are compared, we will use superscripts, e.g. $L_N^{\ell,\ell'}(\cdot,\cdot)$ if we compare configurations $\boldsymbol{\sigma}^{\ell}$ and $\boldsymbol{\sigma}^{\ell'}$.

1.2.2. Order Parameters

Next, we introduce two very important quantities which are generalizations of the q order parameter in the SK model. Like that one, they are defined as solutions of fixed point equations. We explain their form afterwards.

Given the quantity $\kappa\colon \Sigma^2 \to [0,1]$ we define for $s,s' \in \Sigma$:

$$\pi(s) := \sum_{s''\in\Sigma}\kappa(s,s'')$$
$$\Phi_\kappa(s) := \tfrac{1}{2}\gamma(s,\pi) - \tfrac{1}{2}\Gamma^\star(s,s;\kappa)$$

Now, assume that $0 \overset{\Gamma}{\precsim} \kappa$. Then, let $\big(Y(s)\big)_{s\in\Sigma}$ be a Gaussian field with covariance matrix Γ_κ and define for $s \in \Sigma$:

$$\Pi_\kappa(s) := \frac{1}{Z_\kappa}p(s)\exp\left\{\beta Y(s) + \beta^2\Phi_\kappa(s)\right\}, \quad Z_\kappa := \sum_{s\in\Sigma}p(s)e^{\beta Y(s)+\beta^2\Phi_\kappa(s)}, \tag{1.5}$$

Now, κ is characterized by the following fixed point equation:

Lemma 1.6. *(a) For any Γ and p the equation*

$$\kappa(s,s') = \mathbb{E}_Y \Pi_\kappa(s)\Pi_\kappa(s'), \qquad \forall s,s' \in \Sigma \tag{1.6}$$

does have a fixed point solution κ, s.t. $0 \overset{\Gamma}{\precsim} \kappa$.

(b) There is a $\beta_0(\Gamma,p) > 0$ such that for $\beta < \beta_0(\Gamma,p)$ this solution is unique.

Proof. (a) This is a consequence of Brouwer's fixed-point theorem.

Let $\mathcal{M}_\Gamma$ be the set of all symmetric matrices κ with entries in $[0,1]$, $\sum_{s,s'} \kappa(s,s') = 1$, and s.t. $0 \overset{\Gamma}{\precsim} \kappa$.

This set is non-empty (e.g. $\kappa \equiv \frac{1}{|\Sigma|^2}$), compact, and convex. Indeed if $\kappa, \kappa' \in \mathcal{M}_\Gamma$, $x \in [0,1]$, and $a \in \mathbb{R}^\Sigma$ then

$$\sum_{s,s'} a_s \cdot a'_s \cdot \Gamma^\star(s,s'; x\cdot\kappa + (1-x)\kappa')$$
$$= x \sum_{s,s'} a_s \cdot a'_s \Gamma^\star(s,s';\kappa) + (1-x) \sum_{s,s'} a_s \cdot a'_s \cdot \Gamma^\star(s,s';\kappa') \geq 0$$

because Γ_κ and $\Gamma_{\kappa'}$ are positive semidefinite.

Now given $\kappa \in \mathcal{M}_\Gamma$ the matrix $A(\kappa) := \big(\mathbb{E}_Y \Pi_\kappa(s)\Pi_\kappa(s')\big)_{s,s'}$ is again in $\mathcal{M}_\Gamma$ since:

$$\sum_{s,s'} a_s a_{s'} \mathbb{E}_Y \Pi_\kappa(s)\Pi_\kappa(s') = \mathbb{E}_Y \big(\sum_s a_s \Pi_\kappa(s)\big)^2 \geq 0.$$

Hence $A(\cdot)$ defines a map $A\colon \mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$. This is a continuous map. Actually, in section 5.4 it is proved that it is Lipschitz for any β. And by Brouwer's fixed-point theorem it follows that A has a fixed point in $\mathcal{M}_\Gamma$.

(b) This is proved by the Banach fixed-point theorem – the appropriate Lipschitz constants will be bounded in section 5.4. □

From now on, we choose one of the solutions of (1.6) and call it κ. Actually, it turns out that by the estimate in the proof of *(b)* we are not covering the entire region (1.12).

Now, we let $\pi(s) := \sum_{s'} \kappa(s,s')$. Also, π will be used extensively in the notation. And once we will look at the Parisi formula, π will regain some independence when it will reveal itself as the diagonal of a diagonal matrix corresponding to the self-overlap in the case of d-component spin SK model.

1.2.3. Heuristical Explanation for the Fixed Point Equation

Maybe a heuristic argument can explain the form of Π_κ and the fixed point equations, mainly for the deterministic term $\Phi_\kappa(\cdot)$ that does not appear in the standard SK model fixed point equation. The argument heavily depends on the the assumption that the spins become asymptotically independent (see Theorem 1.9) which is the pivotal feature of the high temperature regime.

The meaning of π and κ becomes much clearer if one recognizes the following implication of Theorem 1.8. We have for all $s, s' \in \Sigma$:

- $\pi(s) = \lim_{N\to\infty} \mathbb{E}\mu_1(s)$ is the marginal distribution of any spin.
- $\kappa(s,s') = \lim_{N\to\infty} \mathbb{E}\mu_1(s)\mu_1(s') = \lim_{N\to\infty} \mathbb{E}\left\langle \delta_i^{1,2}(s,s') \right\rangle$ is the average joint distribution of spins in two replicas.

For this heuristic argument, we will assume this to be the definitions of π and κ and derive the fixed point equation as self-consistency equation.

Comparing these expressions to the fixed point equation in Lemma 1.6 and to the fixed point equation $\pi(s) = \mathbb{E}\Pi_\kappa(s)$, suggests that the probability $\Pi_\kappa(\cdot)$ should be the limit of $\mu_i(\cdot)$ for all $i \in \mathbb{N}$. Therefore, in order to give a heuristics for Π_i and the fixed point equations one has to look deeper into the behavior of μ_i.

In Section 6.2, we will see that under the Gibbs measure, we have for the marginal distribution of the σ_N:

$$\mu_N(t) = \tfrac{p(t)}{Z} \left\langle \exp\left\{ \tfrac{\beta}{\sqrt{N}} \sum_i g_{iN}(\sigma_i, t) \right\} \right\rangle_{N-1} \tag{1.7}$$

$$= \tfrac{p(t)}{Z} \left\langle \exp\left\{ \tfrac{\beta}{\sqrt{N}} \sum_{s\in\Sigma} \sum_i g_{iN}(s,t)\delta_i(s) \right\} \right\rangle_{N-1} \tag{1.8}$$

where by $\langle \cdot \rangle_{N-1}$ we denote the Gibbs expectation w.r.t. the Hamiltonian

$$\frac{\sqrt{N-1}}{\sqrt{N}} \cdot H_{N-1}(\boldsymbol{\sigma}) = \frac{\beta}{\sqrt{N}} \sum_{i<j<N} g_{ij}(\sigma_i, \sigma_j).$$

Therefore, we have to analyze the fluctuation of the local field at σ_i:

$$\frac{\beta}{\sqrt{N}} \sum_{s\in\Sigma} \sum_{i} g_{iN}(s,t)\delta_i(s) = \beta Y_t - \beta X_t \tag{1.9}$$

$$Y_t := \frac{1}{\sqrt{N}} \sum_{s\in\Sigma} \sum_{i} g_{iN}(s,t)\mu_i^{(N-1)}(s)$$

$$X_t := \frac{1}{\sqrt{N}} \sum_{s\in\Sigma} \sum_{i} g_{iN}(s,t)\big(\delta_i(s) - \mu_i^{(N-1)}(s)\big),$$

where $\mu_i^{(N-1)}(s) := \langle\, \delta_i(s)\, \rangle_{N-1}$. We now approximate X_t and Y_t using asymptotic independence.

- First, we consider X_t. As Talagrand states in [16, Section 1.5], the asymptotic independence under $\langle \cdot \rangle_{N-1}$ that we have in Theorem 1.9 indicates there might be enough decorrelation to get a central limit theorem – Theorem 6.1 will give a rigorous statement. Then, assuming the spins indeed form a CLT under this Gibbs measure, this also holds for the sequence $\big(\delta_i(s)\big)_{i<N}$ for each $s \in \Sigma$ and any linear combination of them. For fixed g_{iN} the fluctuation of X_t due to the fluctuation of the configurations in $\langle \cdot \rangle_{N-1}$ should be asymptotically a centered Gaussian with variance:

$$\begin{aligned}
\langle\, X_t^2\, \rangle &= \tfrac{1}{N} \sum_{s,s'} \sum_{i} g_{iN}(s,t) g_{iN}(s',t) \\
&\qquad \cdot \Big\langle\, \big(\delta_i(s) - \mu_i^{(N-1)}(s)\big) \cdot \big(\delta_i(s') - \mu_i^{(N-1)}(s')\big)\, \Big\rangle_{N-1} \\
&= \tfrac{1}{N} \sum_{s,s'} \sum_{i} g_{iN}(s,t) g_{iN}(s',t) \\
&\qquad \cdot \big[\langle\, \delta_i(s)\delta_i(s')\, \rangle_{N-1} - \mu_i^{(N-1)}(s)\mu_i^{(N-1)}(s')\big] \\
&= \tfrac{1}{N} \sum_{s} \sum_{i} g_{iN}(s,t)^2 \mu_i^{(N-1)}(s) \\
&\qquad - \tfrac{1}{N} \sum_{s,s'} \sum_{i} g_{iN}(s,t) g_{iN}(s',t) \mu_i^{(N-1)}(s)\mu_i^{(N-1)}(s')
\end{aligned}$$

  Now by the law of large numbers, and since the $\mu_i^{(N-1)}$ are independent of the g_{iN}, $i < N$, we have:

$$\begin{aligned}
&\tfrac{1}{N}\sum_{s,s'}\sum_{i} g_{iN}(s,t)g_{iN}(s',t)\mu_i^{(N-1)}(s)\mu_i^{(N-1)}(s') \\
&\approx \tfrac{1}{N}\sum_{s,s'}\Gamma(s,t,s',t)\sum_{i}\mu_i^{(N-1)}(s)\mu_i^{(N-1)}(s') \qquad (1.10)\\
&\approx \sum_{s,s'}\Gamma(s,t,s',t)\mathbb{E}\mu_1(s)\mu_1(s') = \Gamma^\star(t,t;\kappa)
\end{aligned}$$

The last approximation again is a law of large numbers for the asymptotically i.i.d. sequence $\mu_i(\cdot)$, $i \geq 1$. Similarly, we have

$$\tfrac{1}{N}\sum_{s}\sum_{i} g_{iN}(s,t)^2\mu_i^{(N-1)}(s) \approx \gamma(t,\pi).$$

Hence X_t behaves under $\langle\,\cdot\,\rangle_{N-1}$ as a Gaussian with variance $\gamma(t,\pi) - \Gamma^\star(t,t;\kappa)$.

- Now, we look at Y_t. Again, since $\mu_i^{(N-1)}$ and g_{iN}, $i < N$, are independent, we look at the covariance of Y_t and $Y_{t'}$ only integrating out all the g_{iN}, $i < N$:

$$\mathbb{E}_{(g_{iN})_{i<N}} Y_tY_{t'} = \tfrac{1}{N}\sum_{s}\Gamma(s,t,s',t')\sum_{i}\mu_i^{(N-1)}(s)\mu_i^{(N-1)}(s') \approx \Gamma_\kappa(t,t') \qquad (1.11)$$

  where we again used the law of large numbers for the asymptotic i.i.d. sequence $\mu_i(\cdot)$, $i \geq 1$. Since this last expression is deterministic, we just assume that the Y_t are Gaussians with this covariance structure.

Therefore:

$$\mu_N(t) = \frac{p(t)}{Z}e^{\beta Y_t}\cdot\left\langle\, e^{-\beta X_t}\,\right\rangle_{N-1} \approx \frac{1}{Z}e^{\beta Y_t + \frac{\beta^2}{2}\left(\gamma(t,\pi)-\Gamma^\star(t,t;\kappa)\right)} \approx \Pi_\kappa(t)$$

Looking at (1.9), we see that the local field at σ_i splits into two parts: the first one due to the disorder $g_{iN}(\mu_i,t)$ and the second one X_t due to the quenched fluctuation of the fellow spins. It might state: My decision making is based partly on the social structure and partly on the decision of my peers. In this light, we also could have defined $\Pi_\kappa(s)$ as:

$$\tfrac{1}{p(s)}Z\mathbb{E}_{\xi(s)}\exp\left[Y(s)+\beta\xi(s)\right],$$

where $\xi(s)$ is a Gaussian with variance $\mathbb{E}\xi_s = \gamma(s,\pi) - \Gamma^\star(s,s;\kappa)$. This means, the deterministic term $\Phi_\kappa(\cdot)$ is just the expectation of the fluctuation of the field at site σ_i due to the quenched fluctuation of the other spins.

1.3. Summary of Results

The main results we have for this system are the following ones – they all assume 'high enough temperature' and are all generalizations of proofs that can be found in the work of Talagrand, precise citations can be found in the proofs we will give in Part II. We will then need a more general setting and actually prove more than is stated here. Therefore, all statements will be repeated in Part II in the refined setting.

By assuming 'high enough temperature' we mean:

$$\beta < \beta_0 = \beta_0(\Gamma, p) := \frac{1}{4\sqrt{\sum_{s,t,s',t'} |\Gamma(s,t,s',t')|}}. \tag{1.12}$$

Throughout this whole thesis, we adopt Talagrand's convention of using a constant L that might be different in each statement, just not dependent on the number of spins N – one can think of it as the maximum of all the needed numbers in all the statements[1]. But in contrast to his convention, we will use the constant K for a very specific value:

$$K := K_\Gamma := \sum_{s,t,s',t'} |\Gamma(s,t,s',t')|. \tag{1.13}$$

1.3.1. The Free Energy

One of the most important quantities that needs to be analyzed in any spin glass model is the the free energy $\frac{1}{N}\log Z_N$, as was indicated already in the beginning. Usually, different aspects of it need different machineries:

- The convergence of the quenched free energy $\frac{1}{N}\mathbb{E}\log Z_N$ is usually handled by superadditivity arguments due to Guerra-Toninelli [7] and [6].

[1] If we are not mistaken, actually $L = 48K_\Gamma$ should be sufficient for the calculations in Chapter 5.

But as we will see in Section 3.1 in more general models, this may or may not be feasible.

- The so-called 'self-averaging' property:

$$\lim_{N\to\infty}\frac{1}{N}\log Z_N = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log Z_N$$

this can be shown by concentration of measure. We have proved this already in [18] and repeat the proof in Theorem 4.9.

- Last, but certainly not least is the calculation of the limit of the free energy. The array of attacks on this problem has grown over the decades, which shows that this is a non-trivial issue.

The main result here is the prove of the formula. That is we show for high enough temperature the convergence of the quenched free energy in the thermodynamic limit $N \to \infty$ to:

$$f(\beta,\Gamma) := \; -\tfrac{\beta^2}{4}\Big[\gamma(\pi,\pi) - \Gamma^\star(\kappa;\kappa)\Big] + \mathbb{E}\log \mathrm{Tr}_s\, e^{\beta Y(s)+\beta^2\Phi_\kappa(s)}$$

Theorem 1.7. *Assume* (1.12). *Then:*

$$\Big|\tfrac{1}{N}\mathbb{E}\log Z_N - f(\beta,\Gamma)\Big| \le \frac{L}{N},$$

Under some assumptions on Γ we were able to prove that even on infinite Σ with Gaussian Random Field interaction, c.f. Chapter 7.

1.3.2. The Gibbs Measure at High Enough Temperature

After knowing the free energy, the question is what we can say about the Gibbs measure in this regime.

First, we can observe that the order parameters we introduced in the previous section do have an intrinsic meaning: They are the limits of the empirical distributions:

Theorem 1.8. *Assume* (1.12). *Then, for all $s, s' \in \Sigma$ we have:*

$$\nu\left[\Big(L_N(s) - \pi(s)\Big)^2\right] \leq \frac{L}{N} \quad \textit{and}$$
$$\nu\left[\Big(L_N^{1,2}(s,s') - \kappa(s,s')\Big)^2\right] \leq \frac{L}{N}.$$

Under some assumptions on Γ we were able to prove the corresponding result also for infinite Σ with Gaussian Random Field interaction, c.f. Chapter 7.

Of course this implies using Jensen's inequality on the $\langle \cdot \rangle$-average in the μ_i terms:

$$\mathbb{E}\left[\tfrac{1}{N}\sum_i \mu_i(s) - \pi(s)\right]^2 \leq \frac{L}{N}, \quad \mathbb{E}\left[\tfrac{1}{N}\sum_i \mu_i(s)\mu_i(s') - \kappa(s,s')\right]^2 \leq \frac{L}{N}$$

This means that the quenched marginals μ_i have a LLN converging to π and the same holds for $\mu_i(\cdot)\mu_i(\cdot)$ converging to κ. Hence π and κ are indeed the expectations as we indicated in the beginning of Section 1.2.3. Furthermore, by the triangle inequality the δ_i also have a quenched LLN.

Next, we get the main characteristic of the high temperature regime, namely the asymptotic independence of the spins $\sigma_1, \sigma_2, \ldots$ as $N \to \infty$ under the Gibbs measure. More precisely, Theorem 1.8 implies the asymptotic convergence of the marginal law of a finite set of spins to its independent law:

Theorem 1.9. *Assume* (1.12) *and fix $n \in \mathbb{N}$. Let $P_{1\cdots n}(s_1, \ldots, s_n)$ be the marginal distribution of the first n spins under the Gibbs distribution $\langle \cdot \rangle$. Let $\boldsymbol{\mu}_n := \mu_1 \otimes \cdots \otimes \mu_n$ be the probability on Σ^n where σ_i has marginal μ_i but the $\sigma_1, \ldots, \sigma_n$ are independent. Then, we have*

$$\mathbb{E}\left[\big(|P_{1\cdots n} - \boldsymbol{\mu}_n|_{\mathrm{TV}}\big)^2\right] \leq \frac{L}{N},$$

where $|\cdot|_{\mathrm{TV}}$ is total variation.

1.3.3. TAP Equations

Knowing Theorem 1.9, the interesting question is to get to know the quenched distribution of the $\mu_i(\cdot)$. Thouless, Anderson and Palmer introduced a set of equations that answer this question. In our setting, the TAP equations are:

Theorem 1.10. *Assume* (1.12) *and let $t \in \Sigma$. Then for all $i \leq N$:*

$$\mathbb{E}\left|\mu_i(t) - \tfrac{p(t)}{Z} \exp\left[\frac{\beta}{\sqrt{N}} \sum_{j \neq i} g_{jN}(\mu_j, t) + \beta^2 \Phi_\kappa(t) - \beta^2 \Psi_\kappa(t, \mu_i)\right]\right| \leq \frac{L}{N},$$

$$Z := \mathrm{Tr}_{t'} \exp\left[\frac{\beta}{\sqrt{N}} \sum_{j \neq i} g_{jN}(\mu_j, t') + \beta^2 \Phi_\kappa(t) - \beta^2 \Psi_\kappa(t', \mu_i)\right]$$

where $\Psi_\kappa(t, t') := \sum_s \pi(s) \Gamma^\star(t, t'; s, s) - \Gamma^\star(t, t'; \kappa)$ is the Onsager term in our setting.

See equation (2.1) to compare this to the standard case.

1.3.4. Conjectures: Parisi Formula and the High Temperature Region

Last but not least, we investigate the low temperature regime. We did not try to do this rigorously and only give conjectures.

In order to state the Parisi formula we consider the set RPC of all triples $(K, \mathbf{Q}, \mathbf{m})$ consisting of the following data:

- a $K \in \mathbb{N}$,
- $\mathbf{Q} = (Q_0, \ldots, Q_{K+1})$, a sequence $0 = Q_0 \overset{\Gamma}{\precsim} Q_1 \overset{\Gamma}{\precsim} \ldots \overset{\Gamma}{\precsim} Q_{K+1} = \mathrm{diag}(\pi)$ of symmetric matrices, with non-negative entries summing up to 1 (Q_0 is exempt from the last condition), where Q_{K+1} is a diagonal matrix with diagonal π, and
- $\mathbf{m} = (m_0, \ldots, m_K)$, a non-decreasing sequence $0 = m_0 \leq m_1 \leq \cdots \leq m_K = 1$.

Then, given $(K, \mathbf{Q}, \mathbf{m}) \in \mathrm{RPC}$, this is used to define a sequence of random variables $Y_1, \ldots, Y_{K+1}$ as follows. First, define the random variable $Y_{K+1} := \mathrm{Tr}_s\, e^{\beta \sum_{i=0}^{K} g_i(s)}$. Here $(g_i)_i$ are i.i.d. Gaussian fields with covariance matrices $\Gamma_{Q_{i+1} - Q_i}$. Then, let recursively:

$$Y_i := \left[\mathbb{E}_{g_i}(Y_{i+1}^{m_i})\right]^{1/m_i},$$

where $\mathbb{E}_{g_i}$ is the integration of just g_i. Remark that Y_i only depends on the variables $g_0, \ldots, g_{i-1}$.

Then, our conjecture for the limit of the free energy at all $\beta \geq 0$ is:

Conjecture 1.11 (Parisi formula)**.**

$$\begin{aligned}&\lim_{N\to\infty}\tfrac{1}{N}\log Z_N\\&=\inf_{(K,\mathbf{Q},\mathbf{m})\in\mathrm{RPC}}\left[\mathbb{E}\log Y_1-\frac{\beta^2}{4}\sum_{i=1}^{K}m_i\big(\Gamma^\star(Q_{i+1};Q_{i+1})-\Gamma^\star(Q_i;Q_i)\big)\right],\end{aligned}$$

Theorem 1.7 says that for $\beta<\beta_0$ the infimum already is attained at $K=1$, more precisely it is attained for:

$$K=1,\qquad \mathbf{Q}=(0,\kappa,\operatorname{diag}(\pi)),\qquad \mathbf{m}=(0,1)$$

where by $\operatorname{diag}(\pi)$ we denote the diagonal matrix having the entries of π on the diagonal. Then $g_0(s)$ is our usual $Y(s)$, as it has covariance matrix $\Gamma_\kappa=\Gamma^\star(\cdot,\cdot;\kappa-0)$. And $g_1(s)$ has variance

$$\Gamma^\star(s,s;\operatorname{diag}(\pi)-\kappa)=\gamma(s,\pi)-\Gamma^\star(s,s;\kappa)=2\Phi_\kappa(s).$$

Hence we have:

$$Y_2=\operatorname{Tr}_s e^{\beta Y(s)+\beta g_1(s)}\qquad\qquad Y_1=\mathbb{E}_{g_1}Y_2=\operatorname{Tr}_s e^{\beta Y(s)+\Phi_\kappa(s)}$$

Therefore, this is another explanation of the deterministic term.

According to [11], this leads to conjectures about the parameters that lead to 'high temperature' behavior. The latter is characterized as the region of the parameter space where our formula for the free energy holds, which is thought to be equivalent to the property of the empirical distribution converging.

Panchenko characterizes this as the region where $K=1$ minimizes the infimum. A necessary condition for this is that the infimum over all $K\le 2$ is attained for $K=1$. Hence, if we define for all $0\le m\le 1$ and all $\kappa\preceq A\preceq\operatorname{diag}(\pi)$

$$\begin{aligned}\Psi(m,A):=&-\tfrac{\beta^2}{4}\big[\gamma(\pi;\pi)-\Gamma^\star(A;A)+m\Gamma^\star(A;A)-m\Gamma^\star(\kappa;\kappa)\big)\\&+\tfrac{1}{m}\mathbb{E}\log\mathbb{E}_1\left[\big(\operatorname{Tr}_s e^{Y'(s)}\big)^m\right],\end{aligned}$$

where

$$\begin{aligned}Y'(s)&:=g_0(s)+g_1(s)+\Phi_{\operatorname{diag}(\pi)-A}(s),\\ \Phi_{\operatorname{diag}(\pi)-A}(s)&=\tfrac{1}{2}\big(\gamma(s,\pi)-\Gamma_A(s,s)\big)\end{aligned}$$

then the Replica symmetric formula is the case where this $\Psi(m, A)$ is maximized at $m = 1$ and $Q = \kappa$. Now consider

$$f(A) := \frac{\partial \Psi(m, A)}{\partial m}\bigg|_{m=1} = -\tfrac{\beta^2}{4}\Big(\Gamma^\star(A; A) - \Gamma(\kappa; \kappa)\Big) \\ + \mathbb{E}\tfrac{\mathrm{Tr}_s\, e^{Y'(s)}}{\mathbb{E}_1\, \mathrm{Tr}_s\, e^{Y'(s)}} \cdot \log\Big[\tfrac{\mathrm{Tr}_s\, e^{Y'(s)}}{\mathbb{E}_1\, \mathrm{Tr}_s\, e^{Y'(s)}}\Big]$$

Then, a necessary condition for $\mathcal{P}(\Gamma) = \mathcal{P}_1(\Gamma)$ is that:

$$f(A) \leq 0 \qquad \text{for all} \qquad \kappa \overset{\Gamma}{\preccurlyeq} A \overset{\Gamma}{\preccurlyeq} P. \tag{1.14}$$

The AT-Line corresponds to the Hessian of f being negative semidefinite at $A = Q$. Already [11] showed that those two conditions are not equivalent in generalized versions of the SK model.

1.4. Structure of the Thesis

Part I is aimed at introducing the model in its whole breadth and to motivate why it is interesting to study. Therefore, we only look at the most prominent version (1.2). In this first chapter, we gave an overview of our model and the results and conjectures we achieved. The next two chapters will be devoted to motivate this model – first by giving several examples in Chapter 2, and then in Chapter 3 by showing in a simple setting what special problems this model poses for the standard methods.

In the second part of this thesis, we become more technical and will consider the diluted version of the model. That is, we first have to revisit the model and redefine it in the more general setting (4.5) based on the class of Hamiltonians method introduced in [15]. This also forces us to restate the theorems. They are afterwards proved in Chapter 5.

In Chapter 6 we will prove a kind of central limit theorem and derive from that the TAP equations. Later, we generalize the multivariate SK model to an infinite dimensional setting, generalizing the d-component spin SK model, where $\Sigma \subset \mathbb{R}^d$ can be infinite. Finally, we will investigate how the famous Parisi formula might look in our setting, describing the free energy at all $\beta \geq 0$.

2. Examples

In this section, we illustrate our model with some examples.

2.1. The SK Model

The **Standard SK model** – introduced by Sherrington and Kirkpatrick in [13] – is, as already stated, one of the most prominent mean field spin glass models and also spawned our model. We will see in the upcoming chapters up to how special the standard SK model is in our framework. Here, we want to quickly show how this case fits into our framework.

Of course the standard SK model is just the case where $\Sigma = \{\pm 1\}$ and $\Gamma(s,t,s',t') = sts't'$. Furthermore, we have $p(s) = \frac{e^{hs}}{2\operatorname{ch}(h)}$. Now the well known formulas and equations follow easily, once on makes the following observation:

$$\Phi(s) = \frac{1}{2}\Big(\sum_{t\in\Sigma} s^2 \cdot t^2 \cdot \pi(t) - \sum_{t,t'\in\Sigma} s^2 tt'\kappa(t,t')\Big) = \frac{1}{2}\big(1-q\big),$$

where

$$q := \sum_{t,t'} tt'\kappa(t,t')$$

is the usual order parameter for the SK model. Hence, $\Phi(s) = \frac{1-q}{2}$ is a constant which does not depend on s and cancels out in all the corresponding expressions. This explains why this deterministic term does not appear in the discussion on the SK model.

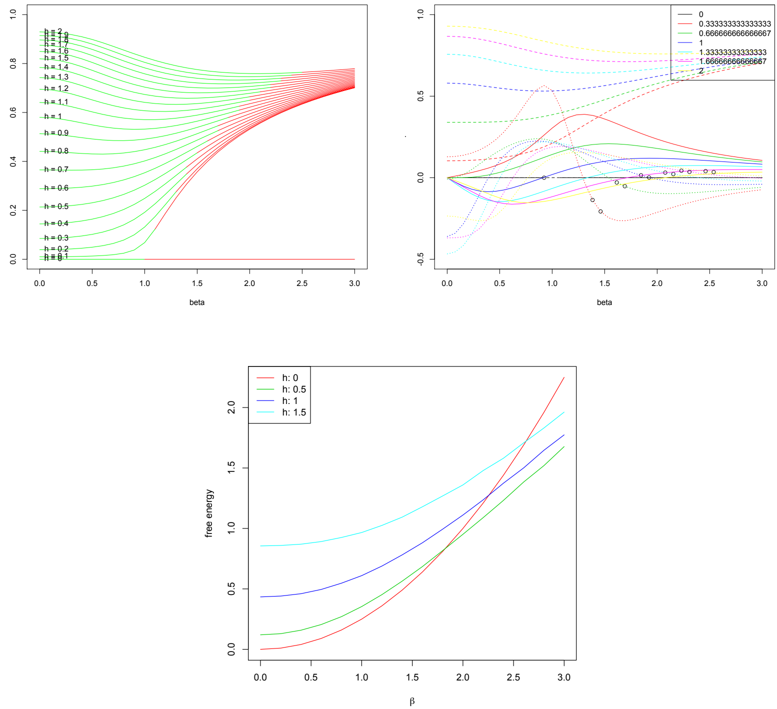

Figure 2.1.: The standard SK model order parameter q as function of β for different parameters h. The first figure shows the value of q in green up to the AT-line and from then on it is red. The second figure shows in solid the first derivative w.r.t. β of q, together with the second derivative in dots, and the original q in dashed lines. The third figure gives the free energy for the SK model as a function of β.

We present the second figure in order to see whether the AT-line corresponds to the inflection points of q. The change of the AT-line is between the circles that we put on the second derivative plots. They do not coincide with the zeros of those curves, therefore the inflection points are not corresponding to the AT-line.

Further, observe further the fixed point equation for q is implied by ours:

$$
\begin{aligned}
q &= \sum_{t,t'} tt'\kappa(t,t') = \mathbb{E}\left(\sum_t t\Pi(t)\right)^2 \\
&= \mathbb{E}\left(\frac{e^{\beta z(1)+\frac{1}{2}(1-q)+h} - e^{\beta z(-1)+\frac{1}{2}(1-q)-h}}{e^{\beta z(1)+\frac{1}{2}(1-q)+h} + e^{\beta z(-1)+\frac{1}{2}(1-q)-h}}\right)^2 = \mathbb{E}\,\mathrm{th}^2(\beta\sqrt{q}z+h),
\end{aligned}
$$

since $\Gamma_\kappa(s,s') = ss'\sum_{t,t'} tt'\kappa(t,t') = ss'q$ is of rank 1 and therefore $z(s) = \sqrt{q}z \cdot s$ for some standard Gaussian z.
The free energy formula is:

$$
\begin{aligned}
& -\tfrac{\beta^2}{4}(1-q^2) + \mathbb{E}\log\tfrac{1}{2}\sum_{s=\pm 1} e^{\beta\sqrt{q}zs+hs+\beta^2\Phi_q(s)} \\
= & -\tfrac{\beta^2}{4}(1-q^2) + \mathbb{E}\log\mathrm{ch}(\beta\sqrt{q}z+h) + \tfrac{\beta^2}{2}[1-q] \\
= & \ \tfrac{\beta^2}{4}(1-q)^2 + \mathbb{E}\log\mathrm{ch}(\beta\sqrt{q}z+h)
\end{aligned}
$$

This is believed to hold up to the so-called AT-line for (β, h), that is those values of $\beta, h \geq 0$ where:

$$
\beta^2\,\mathbb{E}\frac{1}{\mathrm{ch}^4(\beta g\sqrt{q}+h)} < 1
$$

For the SK model, we have $K_\Gamma = 16$, hence our proofs are valid for $\beta < \beta_0 = \frac{1}{4\sqrt{K_\Gamma}} = \frac{1}{16}$. In [16, Chapter 1], this is done for $\beta < \frac{1}{2}$.
Now, from this formula we can derive for instance the average interaction energy by taking the derivative w.r.t. β. For this, first observe:

$$
\tfrac{d}{d\beta}\mathbb{E}_z\log\mathrm{ch}(xz+h) = \mathbb{E}_z\,\mathrm{th}(xz+h)\cdot zx' = xx'\cdot(1-\mathbb{E}_z\,\mathrm{th}^2(xz+h)),
$$

where x' is the derivative of x. Hence:

$$
\begin{aligned}
\tfrac{d}{d\beta}\varphi &= \tfrac{\beta}{2}(1-q)^2 - \tfrac{\beta^2}{2}(1-q)q' + (1-q)\cdot\beta\sqrt{q}(\sqrt{q}+\tfrac{\beta}{2\sqrt{q}}q') \\
&= \tfrac{\beta}{2}(1-q)^2 - \tfrac{\beta^2}{2}(1-q)q' + (1-q)\cdot(\beta q+\tfrac{\beta^2}{2}q') \\
&= \tfrac{\beta}{2}(1-q^2).
\end{aligned}
$$

And for the average magnetization:

$$\begin{aligned}\tfrac{d}{dh}\varphi &= -\tfrac{\beta^2}{2}(1-q)q' + \mathbb{E}_z \operatorname{th}(\beta\sqrt{q}z+h) + \tfrac{\beta^2}{2}(1-q)q' \\ &= \mathbb{E}_z \operatorname{th}(\beta\sqrt{q}z+h).\end{aligned}$$

Now we want to show how the SK model looks seen through the glasses of our framework. First, we have

$$\Gamma = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \Gamma^\star$$

Then, given q and because

$$\Gamma_\kappa = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \sum_{t,t'} tt'\kappa(t,t') = \begin{pmatrix} q & -q \\ -q & q \end{pmatrix}$$

is of rank 1, we have:

$$\kappa(s,s') = \mathbb{E}\frac{1}{Z_q^2} \cdot \exp[\beta\sqrt{q}z \cdot (s+s') + h(s+s')], \qquad Z_q = \operatorname{ch}(\beta\sqrt{q}z+h).$$

For the symmetric case, where $h=0$ and $(s,s')=(1,1)$, we get:

$$a := \kappa(1,1) = \mathbb{E}\frac{1}{(1+e^{-\beta\sqrt{q}z})^2},$$

which by the properties of κ and symmetry of states determine κ completely:

$$\kappa = \begin{pmatrix} a & 1/2-a \\ 1/2-a & a \end{pmatrix}.$$

One further result that we will generalize are the TAP-equations. In the standard SK model they are given by:

$$\mathbb{E}\Big[\langle \sigma_N \rangle - \operatorname{th}\big(\tfrac{\beta}{\sqrt{N}}\sum_{i<N} g_{iN}\langle \sigma_i \rangle + h - \beta^2(1-q)\langle \sigma_N \rangle\big)\Big]^2 \le \frac{1}{N}. \tag{2.1}$$

This follows directly from our formula in Theorem 1.10, since the $\operatorname{th}(\cdot)$ carries the implicit factor ± 1 that has to be stated in the general case as t.

2.2. Further Known Models

2.2.1. SK Model with d-component Spins

The best understood generalization of the SK model is what Talagrand calls the **SK model with d-component spins** cf. e.g. [16, Section 1.12] or [8] and [4], where $\Sigma \subset \mathbb{R}^d$ and p is a probability measure on $\big(\Sigma, \mathcal{B}(\Sigma)\big)$ with some compact support constraint. We will assume that Σ is the closed support of p. Then, this model has the Hamiltonian

$$H(\boldsymbol{\sigma}) = \sum_{i<j} g_{ij} \left\langle\, \sigma_i, \sigma_j \,\right\rangle, \qquad Z_N = \int_{\Sigma^N} e^{H(\boldsymbol{\sigma})}\, p^{\otimes N}(d\boldsymbol{\sigma})$$

where $\langle\, \cdot, \cdot \,\rangle$ is the standard scalar product in $\mathbb{R}^d$. There are two special cases:

- The case where $\Sigma \subset S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ is called **Heisenberg spin**. This will be the case in the next example.
- The case where $\Sigma \subset [-U, U]$ is called the **Spherical SK model**.

Our framework only applies in the case where p is concentrated on a finite set, so we will assume $\Sigma \subset \mathbb{R}^d$ to be a finite set. For the a ∞-dimensional multivariate SK model, we refer the reader to Chapter 7.

From now on assume that $\Sigma \subset \mathbb{R}^d$ is finite. Then

$$\Gamma(s, t, s', t') = \mathbb{E}\big[g^2 \left\langle\, s, t \,\right\rangle \left\langle\, s', t' \,\right\rangle\big] = \left\langle\, s, t \,\right\rangle \left\langle\, s', t' \,\right\rangle$$

which of course is a rank 1 matrix. The first thing is to calculate the order parameter κ. For this, we first evaluate:

$$\begin{aligned}\Gamma_\kappa(s, s') &= \sum_{t,t'} \kappa(t, t') \left\langle\, s, t \,\right\rangle \left\langle\, s', t' \,\right\rangle = \sum_{u,v=1}^{d} \sum_{t,t'} \kappa(t, t') s_u t_u s'_v t'_v \\ &= \sum_{u,v} s_u s'_v q_{uv} = s^T \mathbf{Q} s',\end{aligned}$$

where $\mathbf{Q} = (q_{uv})_{uv}$ with $q_{uv} := \sum_{t,t'} t_u t'_v \kappa(t, t')$ for all $u, v \leq d$. Let $Y_1, \ldots, Y_d$ be a Gaussian field with covariance matrix $\mathbf{Q}$. Then $Y(s) = \sum_u s_u Y_u$. Further:

$$\begin{aligned}
\Phi_\kappa(s) &= \tfrac{1}{2}\Big[\gamma(s,\pi) - \Gamma^\star(s,s;\kappa)\Big] \\
&= \tfrac{1}{2}\Big[\sum_t \langle\, s,t\,\rangle^2\,\pi(t) - \sum_{t,t'} \langle\, s,t\,\rangle\,\langle\, s,t'\,\rangle\,\kappa(t,t')\Big] \\
&= \tfrac{1}{2}\sum_{u,v} s_u s_v \Big[\rho_{uv} - q_{uv}\Big] = \tfrac{1}{2} s^T(\mathbf{R}-\mathbf{Q})s,
\end{aligned}$$

where $\mathbf{R} = (\rho_{uv})_{uv}$ with $r_{uv} := \sum_t t_u t_v \pi(t)$, $u,v \le d$. Then, our fixed point equation is:

$$\begin{aligned}
\kappa(s,s') = \mathbb{E}\,&\tfrac{p(\{s\})p(\{s'\})}{Z_\kappa^2} \\
&\cdot \exp\Big[\beta\sum_u (s_u + s'_u)Y_u + \tfrac{\beta^2}{2} s^T(\mathbf{R}-\mathbf{Q})s + \tfrac{\beta^2}{2}(s')^T(\mathbf{R}-\mathbf{Q})s'\Big]
\end{aligned}$$

and the free energy for $\beta < \beta_0(\Gamma,p)$ is

$$-\tfrac{\beta^2}{4}\sum_{u,v}\left[\rho_{uv}^2 - q_{uv}^2\right] + \mathbb{E}\log \mathrm{Tr}_s \exp\Big[\beta\sum_u s_u Y_u + \tfrac{\beta^2}{2}s^T(\mathbf{R}-\mathbf{Q})s\Big].$$

Further, the convergence of $L_N(s,s')$ to $\kappa(s,s')$ gives for this model:

$$\begin{aligned}
\rho_{uv} &= \sum_t t_u t_v \pi(t) = \sum_t t_u t_v L_N(t) + o(1) = \tfrac{1}{N}\sum_t t_u t_v \sum_i \delta_i(t) + o(1) \\
&= \tfrac{1}{N}\sum_i \sigma_{i,u}\sigma_{i,v} + o(1) = R_N^{u,v}(\boldsymbol{\sigma}) + o(1) \\
q_{uv} &= \sum_{t,t'} t_u t'_v \kappa(t,t') = \sum_{t,t'} t_u t'_v L_N(t,t') + o(1) \\
&= \tfrac{1}{N}\sum_{t,t'} t_u t'_v \sum_i \delta_i(t,t') + o(1) = \tfrac{1}{N}\sum_i \sigma_{i,u}\sigma'_{i,v} + o(1) \\
&= R_N^{u,v}(\boldsymbol{\sigma},\boldsymbol{\sigma}') + o(1),
\end{aligned}$$

where the reader just witnessed the definition of the self overlap $R_N^{u,v}(\boldsymbol{\sigma})$ and of the overlap $R_N^{u,v}(\boldsymbol{\sigma},\boldsymbol{\sigma}')$. This summarizes the comparison of notions in our framework and the common ones for this model.

It is clear that any d-component SK model has Γ of rank 1. But this is not a sufficient condition. For instance the model with

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

is of rank 1, but cannot be realized in any suitable d-component spin model. Hence, the question is: When is a rank 1 Γ representable by a d-component spin model? Assume Γ is of rank 1, hence there exists one and only one $x \in \mathbb{R}^{d^2}$ s.t. $\Gamma = x^T \cdot x$ and $x_1 \geq 0$. We can assume $d = |\Sigma|$. Let X be the $d \times d$-matrix with the entries in $x \in \mathbb{R}^{d^2}$ row by row. Hence Γ can be represented by a d-component spin model iff there are vectors $v_1, \ldots, v_d$ s.t. $X_{ij} = \langle\, v_i, v_j \,\rangle$. This is equivalent to X being symmetric positive semidefinite.

2.2.2. Anisotropic Heisenberg Spin

To get a more realistic model, physicists consider the SK model with so-called Heisenberg spins instead of the usual Ising spins. Those are 3-component spins, as we already saw in section 2.2.1, i.e. $\Sigma \subset S^2 = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ and

$$H_N(\boldsymbol{\sigma}) = \sum_{ij} g_{ij} \left\langle\, \sigma_i, \sigma_j \,\right\rangle .$$

This resembles reality of course much more in magnetic materials than the simple Ising spin. The Heisenberg spin has rotational symmetry.

Now we consider the anisotropic version of the Heisenberg spin, c.f. [9]. Consider i.i.d. symmetric random matrices $\mathbf{D}_{ij}$, then the corresponding Hamiltonian is:

$$H(\boldsymbol{\sigma}) = \tfrac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \left\langle\, \sigma_i, \sigma_j \,\right\rangle + \sqrt{D} \sum_{i<j} \left\langle\, \sigma_i, \mathbf{D}_{ij} \sigma_j \,\right\rangle .$$

The Heisenberg model corresponds to the case $D = 0$. As soon as $D > 0$ this expression will almost surely not have rotational symmetry, at least not in the models we are considering.

Physicist only consider independent uniform variables on the interval $[-1, 1]$ for the entries of $\mathbf{D}$. This does not suit our situation. This is why we assume the entries of $\mathbf{D}$ to be independent Gaussians with variance 1. For simplicity of calculation – as this is only an example section – we also drop the assumption of $\mathbf{D}$ being symmetric. Furthermore, we need to fix a finite subset $\Sigma \subset S^2$, for instance $\Sigma = \{\pm e_i \mid i = 1, 2, 3\}$, and we stipulate $p(s) := \frac{1}{6}$. Then, the covariance matrix is given as:

$$\Gamma(s,t,s',t') = \beta^2 \langle\, s,t \,\rangle \langle\, s',t' \,\rangle + D \langle\, s,s' \,\rangle \langle\, t,t' \,\rangle, \ \gamma(s,t) = \beta^2 \langle\, s,t \,\rangle^2 + D.$$

and $c_{ij} = \frac{1}{N}$. The matrix given by the right term has full rank, and therefore the whole sum has to have full rank as well. Obviously this model is symmetric in the spins so $\pi(s) = \frac{1}{2d} = \frac{1}{6}$ and κ has to look as:

$$\kappa = \begin{array}{cc} & \begin{array}{cccccc} +\mathbf{e_1} & -\mathbf{e_1} & +\mathbf{e_2} & -\mathbf{e_2} & +\mathbf{e_2} & -\mathbf{e_2} \end{array} \\ \begin{array}{c} +\mathbf{e_1} \\ -\mathbf{e_1} \\ +\mathbf{e_2} \\ -\mathbf{e_2} \\ +\mathbf{e_3} \\ -\mathbf{e_3} \end{array} & \begin{pmatrix} q & \bar{q} & & & & \\ \bar{q} & q & & & q' & \\ & & q & \bar{q} & & \\ & & \bar{q} & q & & \\ & q' & & & q & \bar{q} \\ & & & & \bar{q} & q \end{pmatrix} \end{array}$$

Then, we get:

$$\begin{aligned} \Phi(s) &= \tfrac{1}{2}\gamma(s,\pi) - \tfrac{1}{2}\Gamma^\star(s,s;\kappa) \\ &= \tfrac{1}{12}\sum_t (\beta^2 \langle\, s,t \,\rangle^2 + D) - \tfrac{1}{2}\sum_{t,t'} (\beta^2 \langle\, s,t \,\rangle \langle\, s,t' \,\rangle + D \langle\, t,t' \,\rangle)\kappa(t,t') \\ &= \tfrac{\beta^2}{6} + \tfrac{D}{2} - \beta^2(q-\bar{q}) - 3D(q-\bar{q}) = (\beta^2 + 3D)(\tfrac{1}{6} + \bar{q} - q). \end{aligned}$$

This does not depend on s, so the deterministic field will cancel out in all expressions. Then, we have:

$$\begin{aligned} \Gamma(s,s') &= \Gamma^\star(s,s';\kappa) \\ &= \beta^2 \sum_{t,t'} \langle\, s,t \,\rangle \langle\, s',t' \,\rangle \,\kappa(t,t') + D \sum_{t,t'} \langle\, s,s' \,\rangle \langle\, t,t' \,\rangle \,\kappa(t,t') \\ &= \begin{cases} 2\beta^2(q-\bar{q}) + 6D(q-\bar{q}), & s = s' \\ -\big(2\beta^2(q-\bar{q}) + 6D(q-\bar{q})\big), & s = -s' \\ 0, & \langle\, s,s' \,\rangle = 0 \end{cases} \end{aligned}$$

This means that $Y(s)$ and $Y(s')$ are independent, if $\langle\, s,s' \,\rangle = 0$ and else we have:

$$\mathbb{E}Y(s)Y(\pm s) = \pm\alpha^2, \qquad \alpha := \sqrt{2(q-\bar{q})(\beta^2+3D)}$$

Thus, we can just consider three independent standard Gaussians g_1, g_2, g_3 s.t. $Y(\pm e_i) = \pm\alpha g_i$ and obtain:

$$\Pi(\pm e_i) = \frac{\exp(\pm\alpha g_i)}{2\big(\operatorname{ch}(\alpha g_1) + \operatorname{ch}(\alpha g_2) + \operatorname{ch}(\alpha g_3)\big)}$$

Hence, we have the fixed point equations:

$$\begin{aligned} q &= \mathbb{E}\big(\Pi(e_1)\big)^2 = \mathbb{E}\frac{\exp(\pm\alpha 2g_1)}{4\big(\operatorname{ch}(\alpha g_1) + \operatorname{ch}(\alpha g_2) + \operatorname{ch}(\alpha g_3)\big)^2}, \\ \bar{q} &= \mathbb{E}\,\Pi(e_1)\Pi(-e_1) = \mathbb{E}\frac{1}{4\big(\operatorname{ch}(\alpha g_1) + \operatorname{ch}(\alpha g_2) + \operatorname{ch}(\alpha g_3)\big)^2}, \\ q' &= \mathbb{E}\,\Pi(e_1)\Pi(e_2) = \mathbb{E}\frac{\exp\big(\pm\alpha(g_1+g_2)\big)\exp(\pm\alpha 2g_i)}{4\big(\operatorname{ch}(\alpha g_1) + \operatorname{ch}(\alpha g_2) + \operatorname{ch}(\alpha g_3)\big)^2}, \end{aligned}$$

and the limit of the free energy is:

$$(q - \bar{q})(\beta^2 + 3D) + \mathbb{E}\log\big(\operatorname{ch}(\alpha g_1) + \operatorname{ch}(\alpha g_2) + \operatorname{ch}(\alpha g_3)\big).$$

2.3. New Models

2.3.1. Field of Independent Gaussians

We consider the case where $\Sigma = \{\pm 1\}$ and the $g_{ij}(s,t)$ are i.i.d. variables, $i < j$, $s, t \in \Sigma$. We stipulate $p(1) := p(-1) := \frac{1}{2}$. Then, we have of course:

$$\Gamma(s,t,s',t') = \mathbb{E}g_{ij}(s,t)g_{ij}(s',t') = 1_{(s,t)=(s',t')}, \qquad \Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Observe that this has maximal rank, i.e. 4.

We first discuss the $\pi(s)$ and the $\kappa(s,s')$. Of course, by symmetry, we have $\pi(1) = \pi(-1) = \frac{1}{2}$ and there exists $q \in [0,1]$ such that $\kappa(1,1) = \kappa(-1,-1) = q$ and $\kappa(1,-1) = \frac{1}{2} - q =: q'$. Now in order to develop the fixed point equation, we calculate:

$$\Gamma_\kappa = \Gamma^\star \cdot \begin{pmatrix} q \\ q' \\ q' \\ q \end{pmatrix} = \begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&1 \end{pmatrix} \cdot \begin{pmatrix} q \\ q' \\ q' \\ q \end{pmatrix} = \begin{pmatrix} 2q \\ 0 \\ 0 \\ 2q \end{pmatrix} = \begin{pmatrix} 2q & 0 \\ 0 & 2q \end{pmatrix}$$

Hence, the $Y(\cdot)$ are two independent Gaussians of variance $2q$. Then, we have

$$\Phi_\kappa(s) = \tfrac{1}{2}\big(1-2q\big),$$

which does not depend on s anymore, so it cancels out in the fixed point equations:

$$\begin{aligned} q = \kappa(1,1) &= \mathbb{E}\frac{\exp[2\beta\sqrt{2q}g_1]}{(\exp[\beta\sqrt{2q}g_1] + \exp[\beta\sqrt{2q}g_1])^2} \\ &= \mathbb{E}(1+\exp[\beta\sqrt{2q}(g_1-g_2)])^{-2}, \end{aligned}$$

where g_1 and g_2 are two independent standard Gaussians. Then, $\sqrt{2q}(g_1 - g_2)$ is a Gaussian of variance $4q$. Hence we get a fixed point equation for q:

$$q = \mathbb{E}_g \frac{1}{(1+e^{\beta 2\sqrt{q}g})^2}, \tag{2.2}$$

where g is a standard centered Gaussian variable. Then the free energy becomes:

$$-\frac{\beta^2}{4}(1-4q^2) + \mathbb{E}\log \mathrm{Tr}_s \exp\left[\beta\sqrt{2q}g_s\right] + \tfrac{\beta^2}{2}(1-2q)$$

Now we have:

$$\begin{aligned} \mathbb{E}\log(e^{\sqrt{2q}\beta g_1} + e^{\sqrt{2q}\beta g_2}) &= \sqrt{2q}\mathbb{E}g + \mathbb{E}\log(1+e^{\sqrt{2q}\beta(g_2-g_1)}) \\ &= \mathbb{E}_g \log(1+e^{\sqrt{4q}\beta g}) \end{aligned}$$

and therefore get for the free energy:

$$\frac{\beta^2}{4}(1-2q)^2 + \mathbb{E}_g\log(1+e^{2\sqrt{q}\beta g}) - \log 2,$$

at least for $\beta < \frac{1}{8}$.

2.3.2. Potts-type spins

The Potts model, c.f. [12], is defined on a graph with the following interaction:

$$H(\boldsymbol{\sigma}) = \sum_{u\sim v} \delta_{\sigma(u),\sigma(v)},$$

where the summation is over all edges and Σ is just a finite set of colors, say $\{1, \ldots, k\}$. Hence, this counts the edges which lie inside of a monochromatic component.

For our purposes, we can look at the mean field version, where we forget the graph structure and in compensation introduce Gaussians for each monochromatic pairing[1]. Therefore, we look at **Potts-Type spin** interaction:

$$\frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j)$$

with covariance matrix $\Gamma(s,t,s',t') = 1$ if $s = s' = t = t'$ and $\Gamma(s,t,s',t') = 0$ otherwise. Obviously, this matrix has rank k. For $k = 3$, we have the matrix:

$$\Gamma = \begin{array}{cc} & \begin{array}{ccccccccc} \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{21} & \mathbf{22} & \mathbf{23} & \mathbf{31} & \mathbf{32} & \mathbf{33} \end{array} \\ \begin{array}{c} \mathbf{11} \\ \mathbf{12} \\ \mathbf{13} \\ \mathbf{21} \\ \mathbf{22} \\ \mathbf{23} \\ \mathbf{31} \\ \mathbf{32} \\ \mathbf{33} \end{array} & \begin{pmatrix} 1 & & & & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & 0 & & \\ & & & 0 & & & & & \\ & & & & 1 & & & & \\ & & & & & 0 & & & \\ & & 0 & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 1 \end{pmatrix} \end{array} = \Gamma^\star$$

Further we stipulate $p(s) := \frac{1}{k}$.

Now we want to look at the formulas. First, we have once more to find the $\pi(\cdot)$ and $\kappa(\cdot,\cdot)$. Again, because of the symmetry, the $\pi(\cdot)$ are equidistributed, that is $\pi(s) = \frac{1}{k}$ for $s \in \Sigma$. Also, there are q and q' s.t. $\kappa(s,s) = q$ and $\kappa(s,s') = q'$ for $s \neq s'$. Then, $\pi(s) = \frac{1}{k} = q + (k-1)q'$ implies $q' = \frac{1-kq}{k(k-1)}$. Further, we have:

$$\Phi_\kappa(s) = \tfrac{1}{2}(\pi(s) - \kappa(s,s)) = \tfrac{1-kq}{2k}, \qquad \Gamma_\kappa(s,s') = \delta(s,s')\kappa(s,s) = q\delta(s,s')$$

Therefore, $Y(1), \ldots, Y(k)$ are i.i.d. Gaussians with variance q. Because again, $\Phi_\kappa(s)$ does not depend on s, this cancels out in the fixed point equations.

[1] We could have looked also at the covariance matrix $\Gamma(s,t,s',t') = \delta_{s,t}\delta_{s',t'}$. But this gives a rank 1 matrix, and can actually be seen as an instance of the d-component spin model.

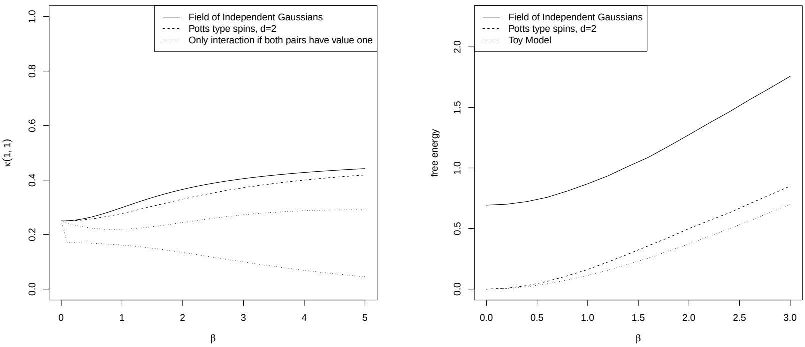

Figure 2.2.: The order parameter $\kappa(1,1)$ (left) and the free energy (right) for three different models. Remark that latter is only rigorous for $\beta < \frac{1}{8}$ for the field of independent Gaussians and for $\beta < \frac{1}{4\sqrt{2}}$ for the Potts-type spin.

Thus, the fixed point equation for q is:

$$q = \mathbb{E}\left(\frac{e^{Y(1)}}{e^{Y(1)}+e^{Y(2)}+\cdots+e^{Y(3)}}\right)^2 = \mathbb{E}\left(\frac{1}{1+e^{\beta\sqrt{2q}g_1}+\cdots+e^{\beta\sqrt{2q}g_{k-1}}}\right)^2$$

and the formula for the free energy:

$$\begin{aligned}
&-\frac{\beta^2}{4}\sum_s(\pi(s)^2-\kappa(s,s)^2)+\mathbb{E}\log\mathrm{Tr}_s\, e^{Y(s)+\beta^2\Phi_\kappa(s)}\\
&=-\frac{\beta^2}{4}(\tfrac{1}{k}-kq^2)+\beta^2\tfrac{1-kq}{2k}+\mathbb{E}\log\mathrm{Tr}_s\, e^{\beta\sqrt{q}g_s}.\\
&=\beta^2\tfrac{(1-kq)^2}{4k}+\mathbb{E}\log\mathrm{Tr}_s\, e^{\beta\sqrt{q}g_s}.
\end{aligned}$$

We proved this for $\beta < \frac{1}{4\sqrt{k}}$.

2.3.3. Toy Model

The following Hamiltonian on $\Sigma = \{\pm 1\}$ also belongs to the realm of our framework, and has some nice properties:

$$H_N(\boldsymbol{\sigma}) := \tfrac{\beta}{2\sqrt{N}} \sum_i \sigma_i \sum_{j \neq i} g_{ij},$$

where the $g_{ij} = g_{ji}$ are i.i.d. standard Gaussians. This indeed fits our setting, if we set:

$$\Gamma := \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \qquad\qquad \Rightarrow \Gamma^\star = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is a rank 1 matrix, but cannot be represented by a d-component model. However, it is given by the interaction:

$$\begin{aligned} g_{ij}(\sigma_i, \sigma_j) &= g_{ij} \cdot \begin{cases} 1 & \text{if } \sigma_i = \sigma_j = 1, \\ 0 & \text{if } \sigma_i \neq \sigma_j, \\ -1 & \text{if } \sigma_i = \sigma_j = -1 \end{cases} \\ &= g_{ij} \cdot \frac{\sigma_i + \sigma_j}{2} \end{aligned}$$

Hence the form of the Hamiltonian.

If we set $p(1) = p(-1)$ again, we have symmetry of states, so $\pi(1) = \pi(-1) = \frac{1}{2}$ and $\kappa(1,1) = \kappa(-1,-1) = q$ and $\kappa(1,-1) = q' := \frac{1}{2} - q$. Derive again the order parameters:

$$\Gamma_\kappa = \begin{pmatrix} q & -q' \\ -q' & q \end{pmatrix}, \qquad \Phi_\kappa(s) = \tfrac{1}{2}[\tfrac{1}{2} - q].$$

Hence the fixed point equation gives:

$$\begin{aligned} q = \kappa(1,1) = \mathbb{E} \frac{e^{2 \cdot \beta Y(1)}}{(e^{\beta Y(1)} + e^{\beta Y(-1)})^2} = \mathbb{E} \frac{1}{(1 + e^{\beta Y(-1) - \beta Y(1)})^2} \\ = \mathbb{E}_g \frac{1}{(1 + e^{\beta g})^2}, \end{aligned}$$

since $\operatorname{Var}\big(Y(-1) - Y(1)\big) = 2q + 2q' = 1$. Surprisingly enough, this does not depend on κ at all, so it is a direct formula for q!

Now the free energy is:

$$
\begin{aligned}
&-\tfrac{\beta^2}{4}\big(\tfrac{1}{2}-2q^2+2(q')^2\big)+\mathbb{E}\log\operatorname{Tr}_s e^{\beta Y(s)}+\tfrac{\beta^2}{2}\big(\tfrac{1}{2}-q\big)\\
&=\tfrac{\beta^2}{2}\big(\tfrac{1}{4}-q+q^2-(q')^2\big)+\mathbb{E}_g\log\frac{1+e^{\beta g}}{2}=\mathbb{E}_g\log\frac{1+e^{\beta g}}{2}.
\end{aligned}
$$

We decided to call this model the **Toy Model** since the free energy actually can be calculated directly:

$$
\begin{aligned}
\tfrac{1}{N}\mathbb{E}\log\operatorname{Tr}_{\boldsymbol{\sigma}} e^{H_N(\boldsymbol{\sigma})}&=\tfrac{1}{N}\mathbb{E}\sum_i\log\operatorname{ch}\left(\tfrac{\beta}{2\sqrt{N}}\sum_{j\neq i}g_{ij}\right)=\mathbb{E}_g\log\operatorname{ch}\left(\tfrac{\beta\sqrt{N-1}}{2\sqrt{N}}g\right)\\
&\xrightarrow[N\to\infty]{}\mathbb{E}_g\log\operatorname{ch}\big(\tfrac{\beta}{2}g\big)
\end{aligned}
$$

which is the same as before. Observe, that this holds for all $\beta\geq 0$, hence this is a very simple model. Actually, one can see from the definition that the spins are just independent conditioned on the disorder.

2.3.4. Compound Models

An interesting feature of our model is that it can have instances of a multivariate SK model as single spins of a super system. If N is a multiple of M then $\Sigma^N=\big(\Sigma^M\big)^{N/M}$ and since Σ^M is a finite set, it is no problem to define a multivariate SK model of $\frac{N}{M}$ spins, where each spin takes values in Σ^M. The question then is: can we somehow relate Z_N to $Z'_{M/N}$?

More technically: consider the consecutive partition of $\{1,\dots,N\}$ into $\hat{N}:=\frac{N}{M}$ clusters of equal size M. Now only look at the bundled interaction between the spins in different clusters i and j:

$$
\hat{H}_{\hat{N}}(\boldsymbol{\tau}):=\tfrac{\beta}{\sqrt{N}}\sum_{i,j=1}^{\hat{N}}\hat{g}_{ij}(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j)=\tfrac{\beta}{\sqrt{N}}\sum_{i,j=1}^{\hat{N}}\sum_{k,l}g_{iM+k,jM+l}(\tau_{ik},\tau_{jl}),
$$

where we denote by τ_{ik} the k-th spin in the i-th cluster. This could of course be done for any multivariate SK model.

Now we confine ourselves to the case where the underlying model is standard

SK model. Then, this gives us the covariance matrix:

$$\Gamma_M(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j,\boldsymbol{\tau}'_i,\boldsymbol{\tau}'_j) = \sum_{k,l} \tau_{ik}\tau_{jl}\tau'_{ik}\tau'_{jl} = M^2 \cdot R_M(\boldsymbol{\tau}_i,\boldsymbol{\tau}'_i)\cdot R_M(\boldsymbol{\tau}_j,\boldsymbol{\tau}'_j),$$
$$\gamma(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j) = 1.$$

Here we used the overlap $R_M(\boldsymbol{\tau},\boldsymbol{\tau}') = \frac{1}{M}\sum_{i\le M}\tau_i\tau'_i$.

We can use this to decompose the partition function, since:

$$H_N(\boldsymbol{\sigma}) = \hat{H}_{\hat{N}}(\boldsymbol{\tau}) + \sqrt{\frac{M}{N}}\sum_i H_M(\boldsymbol{\tau}_i),$$

where we denote by $\boldsymbol{\tau}$ the M-clustering of $\boldsymbol{\sigma}$, i.e. a configuration with each super-spin being an element of $\Sigma_M := \Sigma^M$. Then, the free energy becomes:

$$\begin{aligned}\log Z_N &= \log \mathrm{Tr}_{\boldsymbol{\sigma}\in\Sigma^N}\, e^{H(\boldsymbol{\sigma})} = \log \mathrm{Tr}_{\boldsymbol{\tau}\in\Sigma_M^{\hat{N}}}\, e^{\hat{H}_{\hat{N}}(\boldsymbol{\tau})+\sqrt{\frac{M}{N}}\sum_i H_M(\boldsymbol{\tau}_i)}\\ &= \log\left\langle\, e^{\hat{H}_{\hat{N}}(\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{\hat{N}})}\,\right\rangle^{\otimes\hat{N}} + \hat{N}\cdot\log Z_M.\end{aligned}$$

Here the $\langle\,\cdot\,\rangle^{\otimes N}$ denotes the Gibbs-measure w.r.t. $\sqrt{\frac{M}{N}}H_M$, but with independent copies of the disorder – so the $\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{\hat{N}}$ are not replicas – hence the added notation.

For fixed M and for $\hat{N}$ big enough, $\sqrt{\frac{M}{N}}H_M$ corresponds to the small enough β and therefore will have asymptotically small correlations. Therefore, we might be tempted to forget this average and replace it with some $\hat{\mathrm{Tr}}$ corresponding to $\hat{p}(\mathbf{t}) = p(t_1)\cdots p(t_M)$:

$$\log Z_N = \log \hat{\mathrm{Tr}}_{\boldsymbol{\tau}} e^{\hat{H}_{\hat{N}}(\boldsymbol{\tau}_1,\dots,\boldsymbol{\tau}_{\hat{N}})} + \hat{N}\cdot\log Z_M.$$

Now in order to assess how powerful this might be, we estimate:

$$\begin{aligned}K(\Gamma_M) &= \sum_{\mathbf{s},\mathbf{t},\mathbf{s}',\mathbf{t}'}|\Gamma_M(\mathbf{s},\mathbf{t},\mathbf{s}',\mathbf{t}')| = \left(\sum_{\mathbf{s},\mathbf{s}'}|\sum_{i=1}^M s_i s'_i|\right)^2\\ &= \left(2^M\sum_{\mathbf{s}}|\sum_{i=1}^M s_i|\right)^2 = \left(2^M\sum_{k=1}^M\binom{M}{k}|2k-M|\right)^2 \approx 2^{4M}.\end{aligned}$$

This is of course catastrophically worse then in the case $K(\Gamma_1) = 16$ and stands no chance of being compensated by the factor $\frac{1}{M}$. Maybe it can be done, if there is found a way of approximating $\tilde{\Gamma}_M \approx \Gamma_M$ with $K(\tilde{\Gamma}_M) \leq M$.

3. Peculiarities

Our model exhibits several difficulties and obstacles not present in the standard SK model. The most prominent one is σ^2 not being constant. The issues this poses, have already been studied in the SK model with d-component spins, c.f. 2.2.1.. This was done in several works starting with [5] and culminating in the Thesis [8] and is discussed in Talagrand's book [16, Section 1.12]. This model is handling only some of the Γ at rank 1 as stated above.

Now, in our more general setting – certainly once the rank of Γ becomes bigger than 1 – the major obstacle in fact turns out to be that the covariances of the Hamiltonians do not factor out:

$$\mathbb{E}H_N(\boldsymbol{\sigma})H_N(\boldsymbol{\sigma}') = \tfrac{\beta^2}{N}\sum_{i<j}\Gamma(\sigma_i,\sigma_j,\sigma'_i,\sigma'_j)$$

In the standard SK model, the sum in this expression is just

$$\begin{aligned}\sum_{i<j}\sigma_i\sigma_j\sigma'_i\sigma'_j = \sum_{i<j}\sigma_i\sigma'_i\sigma_j\sigma'_j &= \tfrac{1}{2}\left(\sum_i\sigma_i\sigma'_i\right)\left(\sum_i\sigma_i\sigma'_i\right) - \frac{N}{2}\\ &= \tfrac{N^2}{2}R_N(\boldsymbol{\sigma},\boldsymbol{\sigma}')^2 - \frac{N}{2}.\end{aligned}$$

Even in the SK model with d-component spins, which corresponds to

$$\Gamma(s,t,s',t') = \langle\ s,t\ \rangle\cdot\langle\ s',t'\ \rangle$$

that is the product of the scalar products, we have:

$$\begin{aligned}\tfrac{N}{\beta^2}\mathbb{E}H_N(\boldsymbol{\sigma})H_N(\boldsymbol{\sigma}') &= \sum_{i<j}\langle\ \sigma_i,\sigma_j\ \rangle\cdot\langle\ \sigma'_i,\sigma'_j\ \rangle = \sum_{i<j}\sum_{x,y=1}^{d}\sigma_{i,x}\sigma_{j,x}\sigma'_{i,y}\sigma'_{j,y}\\ &= \tfrac{1}{2}\sum_{x,y=1}^{d}\Bigg(\sum_i\sigma_{i,x}\sigma'_{i,y}\Bigg)\Bigg(\sum_j\sigma_{j,x}\sigma'_{j,y}\Bigg) - \tfrac{1}{2}\sum_{x,y=1}^{d}\sum_i(\sigma_{i,x}\sigma'_{i,y})^2\end{aligned}$$

This does not work in our more general situation. In our case, we have to resort to a workaround, using the empirical measures defined in Chapter 1:

$$
\begin{aligned}
&\tfrac{N}{\beta^2}\mathbb{E}H_N(\boldsymbol{\sigma}^1)H_N(\boldsymbol{\sigma}^2) = \sum_{i<j}\Gamma(\sigma_i^1,\sigma_j^1,\sigma_i^2,\sigma_j^2)\\
&= \sum_{i<j}\sum_{s,t,s',t'}\Gamma(s,t,s',t')\delta_i^{1,2}(s,s')\delta_j^{1,2}(t,t')\\
&= \tfrac{N^2}{2}\sum_{s,t,s',t'}\Gamma(s,t,s',t')L_N^{1,2}(s,s')L_N^{1,2}(t,t') - \tfrac{N}{2}\sum_{s,s'}\Gamma^\star(s,s';s,s')L_N^{1,2}(s,s')\\
&= \tfrac{N^2}{2}\Gamma^\star(L_N^{1,2};L_N^{1,2}) + O(N)
\end{aligned}
$$

We will use this device extensively in the proofs for our model. To illustrate how most of the computations will be done, we now will perform an exemplary calculation for the free energy. This will lead to Theorem 3.2, that is in some cases even stronger than 3.3, which has been proved in [7].

3.1. Superadditivity

It is a natural idea to try to prove at least the a.s. existence of the free energy at all temperatures using the famous Guerra-Toninelli interpolation and superadditivity argument as it was introduced in [6] and perfected in [7]. We will call the general instrument the *smart path method*, as is done in Talagrand's work.

3.1.1. The Smart Path Method

If we want to compare two systems, we define a path along some interpolation parameter $x \in [0,1]$. This gives us e.g. the free energy $f(x)$ of the interpolated system, that will be assumed differentiable on $[0,1]$. Then, we have by the fundamental theorem of calculus:

$$f(1) = f(0) + \int_0^1 f'(x)\,dx \le f(0) + \sup_{x\in[0,1]} f'(x),$$

as well as $f(1) \ge f(0) + \inf_{x\in[0,1]} f'(x)$. These inequalities are most of the time powerful enough, because we can indeed calculate some bound on $f'(x)$, since this term will in all our instances be the expectation of some expression with Gaussians, so that we can apply integration by parts:

$$\mathbb{E}g \cdot f(g_1, \dots, g_n) = \sum_{i=1}^{n} \mathbb{E}(gg_i) \cdot \mathbb{E}\frac{\partial f(g_1, \dots, g_n)}{\partial g_i}$$

For the moment, fix two numbers N and M. We introduce the 'smart path' interpolating between two independent systems with N and M spins and the full system of $N + M$ spins:

$$\begin{aligned} H_x(\boldsymbol{\sigma}) := \beta\sqrt{\tfrac{x}{N+M}} &\sum_{i<j} g_{ij}(\sigma_i, \sigma_j) \\ &+ \beta\sqrt{\tfrac{1-x}{N}} \sum_{i<j\leq N} g'_{ij}(\sigma_i, \sigma_j) + \beta\sqrt{\tfrac{1-x}{M}} \sum_{N<i<j} g'_{ij}(\sigma_i, \sigma_j), \end{aligned}$$

where $(g_{ij})_{i<j\leq N+M}$, $(g'_{ij})_{i<j\leq N}$ and $(g'_{ij})_{N<i<j}$ are i.i.d. sequences of Gaussian fields with covariance matrix Γ mutually independent. Then, we define:

$$\varphi(x) := \mathbb{E} \log \mathrm{Tr}_{\boldsymbol{\sigma}\in\Sigma^{N+M}}\, e^{H_x(\boldsymbol{\sigma})}$$

Notice the absence of a rescaling factor $\frac{1}{N+M}$ in this definition. Observe:

$$\varphi(1) = \mathbb{E} \log Z_{N+M}, \qquad \varphi(0) = \mathbb{E} \log Z_N + \mathbb{E} \log Z_M.$$

Then, we define $\langle \cdot \rangle_x$ as the corresponding Gibbs measure by writing for any function f in n replicas:

$$\begin{aligned} \langle\, f \,\rangle_x &:= \tfrac{1}{Z_x^n} \mathrm{Tr}_{\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n\in\Sigma^N}\, f(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^l) e^{\sum_{\ell=1}^{n} H_x(\boldsymbol{\sigma}^\ell)}, \\ Z_x &:= \mathrm{Tr}_{\boldsymbol{\sigma}\in\Sigma^N}\, e^{H_x(\boldsymbol{\sigma})} \end{aligned}$$

and

$$\nu_x(f) := \mathbb{E}\big[\, \langle\, f \,\rangle_x \big].$$

Then, as we will do often in this work, we continue by differentiating this and integrating by parts.

Differentiation First we differentiate $\varphi(x)$ w.r.t. x:

$$\varphi'(x) = \mathbb{E}\frac{1}{Z_x} \cdot \mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{H_x(\boldsymbol{\sigma})} \cdot H_x'(\boldsymbol{\sigma}) = \mathbb{E}\frac{e^{H_x(\boldsymbol{\sigma})}}{Z_x} H_x'(\boldsymbol{\sigma}) = \nu_x\big[H_x'(\boldsymbol{\sigma})\big],$$

where $H_x'(\boldsymbol{\sigma})$ is the derivative of $H_x(\boldsymbol{\sigma})$ w.r.t. x:

$$\begin{aligned} H_x'(\boldsymbol{\sigma}) = \tfrac{\beta}{2\sqrt{x(N+M)}} \sum_{i<j} g_{ij}(\sigma_i,\sigma_j) - \tfrac{\beta}{2\sqrt{(1-x)N}} &\sum_{i<j\le N} g_{ij}'(\sigma_i,\sigma_j) \\ - \tfrac{\beta}{2\sqrt{(1-x)M}} &\sum_{N<i<j} g_{ij}'(\sigma_i,\sigma_j) \end{aligned}$$

Hence this gives:

$$\begin{aligned} \varphi'(x) = \tfrac{\beta}{2\sqrt{x(N+M)}} \sum_{i<j} \nu_x[g_{ij}(\sigma_i,\sigma_j)] - \tfrac{\beta}{2\sqrt{(1-x)N}} &\sum_{i<j\le N} \nu_x[g_{ij}'(\sigma_i,\sigma_j)] \\ - \tfrac{\beta}{2\sqrt{(1-x)M}} &\sum_{N<i<j} \nu_x[g_{ij}'(\sigma_i,\sigma_j)] \end{aligned} \tag{3.1}$$

Observe that we have an expectation of linear combinations of Gaussians.

Integration by parts If we want to use integration by parts in

$$\begin{aligned} \nu_x[g_{ij}(\sigma_i,\sigma_j)] = \mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathbb{E} g_{ij}(\sigma_i,\sigma_j) \cdot \frac{e^{H_x(\boldsymbol{\sigma})}}{Z_x} = \mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathbb{E}\Bigg[g_{ij}(\sigma_i,\sigma_j) \\ \cdot \frac{e^{\beta\sqrt{\frac{x}{N+M}}\sum g_{ij}(\sigma_i,\sigma_j)+\beta\sqrt{\frac{1-x}{N}}\sum g_{ij}'(\sigma_i,\sigma_j)+\beta\sqrt{\frac{1-x}{M}}\sum g_{ij}'(\sigma_i,\sigma_j)}}{\mathrm{Tr}_{\boldsymbol{\sigma}'}\, e^{\beta\sqrt{\frac{x}{N+M}}\sum g_{ij}(\sigma_i',\sigma_j')+\beta\sqrt{\frac{1-x}{N}}\sum g_{ij}'(\sigma_i',\sigma_j')+\beta\sqrt{\frac{1-x}{M}}\sum g_{ij}'(\sigma_i',\sigma_j')}} \Bigg] \end{aligned}$$

we have to first identify all Gaussians in the fraction that are not independent of $g_{ij}(\sigma_i,\sigma_j)$. Luckily, the g_{ij}' are independent as are all $g_{i'j'}(\cdot,\cdot)$ for $i \neq i'$ or $j \neq j'$. Hence, there is $g_{ij}(\sigma_i,\sigma_j)$ in the Hamiltonian in the numerator and $g_{ij}(\sigma_i',\sigma_j')$ in the Hamiltonian of every summand in the denominator. Thus the derivatives of the fraction are:

$$\begin{aligned}
\frac{\partial}{\partial g_{ij}(\sigma_i,\sigma_j)}\,\frac{e^{H_x(\boldsymbol{\sigma})}}{\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')}} &= \frac{e^{H_x(\boldsymbol{\sigma})}\cdot\beta\sqrt{\frac{x}{N+M}}}{\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')}},\\
\frac{\partial}{\partial g_{ij}(\sigma_i',\sigma_j')}\,\frac{e^{H_x(\boldsymbol{\sigma})}}{\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')}} &= -\frac{e^{H_x(\boldsymbol{\sigma})}}{(\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')})^2}\cdot\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')}\cdot\beta\sqrt{\frac{x}{N+M}}\\
&= -\,\mathrm{Tr}_{\boldsymbol{\sigma}'}\,\frac{e^{H_x(\boldsymbol{\sigma})+H_x(\boldsymbol{\sigma}')}}{(\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')})^2}\cdot\beta\sqrt{\frac{x}{N+M}}
\end{aligned}$$

For the novice reader, we have to point out the emergence of a replica as an effect of derivation. Now, we can perform integration by parts:

$$\begin{aligned}
\nu_x[g_{ij}(\sigma_i,\sigma_j)] &= \beta\sqrt{\tfrac{x}{N+M}}\mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\,\gamma(\sigma_i,\sigma_j)\cdot\frac{e^{H_x(\boldsymbol{\sigma})}}{\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')}}\\
&\quad-\beta\sqrt{\tfrac{x}{N+M}}\mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}\,\Gamma(\sigma_i,\sigma_j,\sigma_i,\sigma_j{}')\cdot\frac{e^{H_x(\boldsymbol{\sigma})+H_x(\boldsymbol{\sigma}')}}{(\mathrm{Tr}_{\boldsymbol{\sigma}'}\,e^{H_x(\boldsymbol{\sigma}')})^2}\\
&= \beta\sqrt{\tfrac{x}{N+M}}\nu_x\left[\gamma(\sigma_i,\sigma_j)-\Gamma(\sigma_i,\sigma_j,\sigma_i,\sigma_j{}')\right]
\end{aligned}$$

Similar expressions hold for the two other summands in (3.1). Hence we have:

$$\begin{aligned}
\varphi'(x) = \tfrac{\beta^2}{2}\nu_x\Bigg[&\tfrac{1}{N+M}\sum_{i<j}\Big(\gamma(\sigma_i,\sigma_j)-\Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j')\Big)\\
&-\tfrac{1}{N}\sum_{i<j\le N}\Big(\gamma(\sigma_i,\sigma_j)-\Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j')\Big)\\
&-\tfrac{1}{M}\sum_{N<i<j}\Big(\gamma(\sigma_i,\sigma_j)-\Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j')\Big)\Bigg]
\end{aligned}$$

Now we apply the device of using empirical distributions, as they were introduced in the beginning of this chapter

$$
\begin{aligned}
\varphi'(x) = \tfrac{\beta^2}{2} \sum_{st} \gamma(s,t) \cdot \nu_x \Bigg[& \sum_{i<j} \frac{\delta_i(s)\delta_j(t)}{N+M} \\
& - \sum_{i<j\le N} \frac{\delta_i(s)\delta_j(t)}{N} - \sum_{N<i<j} \frac{\delta_i(s)\delta_j(t)}{M} \Bigg] \\
& - \tfrac{\beta^2}{2} \sum_{s,t,s',t'} \Gamma(s,t,s',t') \cdot \nu_x \Bigg[\sum_{i<j} \frac{\delta_i^{1,2}(s,s')\delta_j^{1,2}(t,t')}{N+M} \\
& \quad - \sum_{i<j\le N} \frac{\delta_i^{1,2}(s,s')\delta_j^{1,2}(t,t')}{N} - \sum_{N<i<j} \frac{\delta_i^{1,2}(s,s')\delta_j^{1,2}(t,t')}{M} \Bigg]
\end{aligned}
$$

This device factorizes all the summands in the sums over increasing, pairs and so they can be written as squares of the sums, up to insertion of a diagonal term:

$$
\begin{aligned}
\varphi'(x) = \tfrac{\beta^2}{4} \sum_{st} \gamma(s,t) \cdot \nu_x \Bigg[& (N+M) L_{N+M}(s) L_{N+M}(t) \\
& - N L_N^1(s) L_N^1(t) - M L_M^2(s) L_M^2(t) \Bigg] \\
- \tfrac{\beta^2}{2} \sum_{s,t,s',t'} \Gamma(s,t,s',t') \cdot \nu_x \Bigg[& (N+M) \cdot L_{N+M}(s,s') L_{N+M}(t,t') \\
& - N \cdot L_N^1(s,s') L_N^1(t,t') - M \cdot L_M^2(s,s') L_M^2(t,t') \Bigg] + o(N+M),
\end{aligned}
$$

where $L_k(s) = \frac{1}{k}\sum_{i=1}^k \delta_i(s)$ and $L_k(s,s') = \frac{1}{k}\sum_{i=1}^k \delta_i(s,s')$ are the empirical measures and the variants with superscripts cover the spins for $1,\ldots,N$ and $N+1,\ldots,N+M$ respectively. The insertion of the diagonal terms is covered by the $o(N+M)$.

Now, using our notation, we rewrite this as:

$$
\begin{aligned}
\varphi'(x) = \tfrac{\beta^2}{4}\cdot \nu_x \Bigg[&(N+M)\gamma(L_{N+M},L_{N+M}) - N\gamma(L_N^1,L_N^1) - M\gamma(L_M^2 L_M^2) \\
&- (N+M)\,\Gamma^\star(L_{N+M};L_{N+M}) + N\,\Gamma^\star(L_N^1;L_N^1) + M\,\Gamma^\star(L_M^2;L_M^2)\Bigg] \\
&+ o(N+M).
\end{aligned}
$$

Quadratic Form Consider the quadratic form:

$$
Q_\Gamma(f) := \sum_{s,t,s',t'} f(s,s')f(t,t')\cdot\big[\gamma(s,t) - \Gamma(s,t,s',t')\big]
$$

for all symmetric $|\Sigma|\times|\Sigma|$-matrices f. Observe that in this definition of $Q_\Gamma(f)$, the first summand of the bracket does not depend on s' and t'.

Then, the above equation can be rewritten as:

$$
\begin{aligned}
\varphi'(x) = (N+M)\tfrac{\beta^2}{4}\cdot\nu_x\Big[Q_\Gamma(L_{N+M}) - \tfrac{N}{N+M}Q_\Gamma(L_N^1) - \tfrac{M}{N+M}Q_\Gamma(L_M^2)\Big] \\
+ o(1)
\end{aligned}
$$

Hence, using the fundamental theorem of calculus, we proved the following

Lemma 3.1. *Let $x = \frac{N}{N+M}$ and $\bar{x} = 1-x$. Then we have:*

$$
\begin{aligned}
\mathbb{E}\log Z_{N+M} \geq \mathbb{E}\log Z_N + \mathbb{E}\log Z_M + \tfrac{\beta^2}{4}\cdot(N+M)\inf_t \nu_t\big[R_{N,M}\big] \\
+ o(N+M), \\
R_{N,M} := Q_\Gamma(L_{N+M}) - xQ_\Gamma(L_N^1) - \bar{x}Q_\Gamma(L_M^2)
\end{aligned}
$$

In the case that the remainder term $R_{N,M}$ is non-negative, this gives superadditivity of $\mathbb{E}\log Z_N$ and hence convergence of $\frac{1}{N}\mathbb{E}\log Z_N$. Thus we have to understand under which circumstances this applies.

3.1.2. Sufficient Condition for Superadditivity

Theorem 3.2. *Assume* Γ *is such that:*

$$0 \geq Q_\Gamma(f) = \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \left[\gamma(s,t) - \Gamma(s,t,s',t')\right] \tag{3.2}$$

for all symmetric $|\Sigma| \times |\Sigma|$*-matrices* f*, s.t.* $\sum_{s,s'} f(s,s') = 0$*. Then the quenched free energy* $\frac{1}{N}\mathbb{E}\log Z_N$ *is superadditive in* N *and hence converges for all* β*.*

If $Q_\Gamma(f) = 0$ *for all such* f*, then the free energy is additive, up to an error vanishing in the thermodynamic limit.*

Proof. Observe that $L_{N+M} = xL_N^1 + \bar{x}L_M^2$ is just a linear interpolation. Therefore the remainder term $R_{N,M}$ in Lemma 3.1 is just the convexity error of the quadratic form $Q_\Gamma(f)$, when we interpolate at $x = \frac{N}{N+M}$ between L_N^1 and L_M^2. Consider the matrix B that defines the quadratic form:

$$B_{(s,s'),(t,t')} = \gamma(s,t) - \Gamma(s,t,s',t').$$

Then, the convexity error of $Q_\Gamma(\cdot)$ can be calculated in the following way:

$$\begin{aligned} Q_\Gamma(xf + \bar{x}g) - xQ_\Gamma(f) - \bar{x}Q_\Gamma(g) & \\ &= (xf + \bar{x}g)^T B(xf + \bar{x}g) - xQ_\Gamma(f) - \bar{x}Q_\Gamma(g) \\ &= (x^2 - x)Q_\Gamma(f) + 2x\bar{x}f^T Bg + (\bar{x}^2 - \bar{x})Q_\Gamma(g) \\ &= -x\bar{x} \cdot \left(Q_\Gamma(f) - 2f^T Bg + Q_\Gamma(g)\right) \\ &= -x\bar{x} \cdot Q_\Gamma(f - g). \end{aligned}$$

Remark that in this calculation, f and g are both symmetric matrices with entries in $[0,1]$, adding up to 1. Therefore $f - g$ is a symmetric matrix with entries in $[-1,1]$ adding up to 0. Therefore, if $Q_\Gamma(\cdot)$ fulfills our assumption, the remainder $R_{N,M}$ is non-negative a.s. $\square$

3.1.3. Guerra-Toninelli and the Local Partition Function

Guerra-Toninelli gave in [7] sufficient conditions for superadditivity for a very broad range of spin glass models, including our generalized version of the SK model:

Theorem 3.3 (Guerra-Toninelli)**.** *Assume that the covariances of the Hamiltonians* $c_N(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := \mathbb{E} H_N(\boldsymbol{\sigma}) H_N(\boldsymbol{\sigma}')$*,* $N \in \mathbb{N}$*, can be written as*

$$c_N(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = N \cdot f\Big(R_N^1(\boldsymbol{\sigma}, \boldsymbol{\sigma}'), \ldots, R_N^k(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\Big) + O(1),$$

where f *is a convex function with continuous derivatives, and where the variables* R_N^i *satisfy:*

$$\begin{aligned} R_N^i(\sigma, \sigma) &\leq M \\ (N+M)\, R_{N+M}^i(\boldsymbol{\sigma}, \boldsymbol{\sigma}') &= N R_N^i(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}'^{(1)} + M R_M^i(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}'^{(2)}), \end{aligned}$$

for all $i, N, M, \boldsymbol{\sigma}, \boldsymbol{\sigma}'$ *where* $\boldsymbol{\sigma}^{(1)}$ *and* $\boldsymbol{\sigma}^{(2)}$ *are the restrictions to the first* N *and last* M *spins, respectively. Then superadditivity holds for the quenched free energy* $\frac{1}{N}\mathbb{E} \log Z_N$*.*

For the further conditions that have to be applied in other models, we refer the reader to [7]. In our setting, as we described in the beginning of this chapter, we have:

$$c_N(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \tfrac{N}{2}\Gamma^\star(L_N^{1,2}; L_N^{1,2}) - \tfrac{1}{2}\sum_{s,s'} \Gamma(s, s, s', s') L_N^{1,2}(s, s')$$

Thus this fits easily into the setting of the above theorem. Hence superadditivity is given, if $\Gamma^\star$ is positive semidefinite. Maybe it is just a coincidence that, in the physicist's Replica Trick calculation, positive semidefiniteness of $\Gamma^\star$ has to be assumed as well.

The idea behind that approach is to constrain the partition function only to similar configurations $\boldsymbol{\sigma}$, i.e. such $\boldsymbol{\sigma}$ where the empirical distribution L_N falls into an $\frac{\varepsilon}{2}$-neighborhood $V = V(\boldsymbol{\eta}, \frac{\varepsilon}{2})$ of some configuration $\boldsymbol{\eta} \in \Sigma^N$. This means that for all $\boldsymbol{\sigma}, \boldsymbol{\sigma}' \in L_N^{-1}(V)$ we have

$$|L_N^{\boldsymbol{\sigma}}(\cdot) - L_N^{\boldsymbol{\sigma}'}(\cdot)| \leq \varepsilon.$$

Therefore, in that case, one can bound the first summand in $Q_\Gamma(L_N^{\boldsymbol{\sigma}} - L_N^{\boldsymbol{\sigma}'})$, since $f := L_N^{\boldsymbol{\sigma}} - L_N^{\boldsymbol{\sigma}'}$ is small. This gives:

$$Q_\Gamma(f) \leq K\varepsilon^2 - \sum_{s,t,s',t'} f(s,s') f(t,t') \Gamma(s,t,s',t') = K\varepsilon^2 - f^T \Gamma^\star f$$

Hence, local almost superadditivity holds under the assumption that $\Gamma^\star$ is positive semidefinite. Then, the important step then in [7] is, to get from this local superadditivity also global superadditivity.

Interesting enough, both Theorems 3.2 and 3.3 cover different cases, depending on Γ. In the following examples, we explore this difference.

3.1.4. Examples

Here already the case $|\Sigma| = 2$ presents several interesting examples. But first observe that for $|\Sigma| = 2$ the assumptions on $f(\cdot,\cdot)$ lead to the following form:

$$f(\cdot,\cdot) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad b := -\frac{a+c}{2}, \qquad a, c \in [-1, 1]. \tag{3.3}$$

SK Model To compare this with the situation in the standard SK model we observe:

$$\begin{aligned} Q_\Gamma(f) &= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \left[\gamma(s,t) - \Gamma(s,t,s',t')\right] \\ &= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \left[1 - sts't'\right] = 0 - \left(\sum_{s,s'} f(s,s')ss'\right)^2 \end{aligned}$$

Which obviously is non-positive. Of course $\Gamma^\star = \Gamma$ is positive semidefinite, and hence Theorem 3.3 can be applied as well.

SK Model with d-component spins We let $\Gamma(s,t,s',t') = \langle\, s,t \,\rangle \langle\, s',t' \,\rangle$. Then this gives:

$$\begin{aligned}
Q_\Gamma(f) &= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \big[\gamma(s,t) - \Gamma(s,t,s',t')\big] \\
&= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \big[\langle\, s,t \,\rangle^2 - \langle\, s,t \,\rangle \langle\, s',t' \,\rangle)\big] \\
&= \sum_{i,j=1}^{d} \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \big[s_i s_j t_i t_j - s_i s'_j t_i t'_j\big] \\
&= \sum_{i,j=1}^{d} \left[\left(\sum_s s_i s_j \sum_{s'} f(s,s') \right)^2 - \left(\sum_{s,s'} s_i s'_j f(s,s') \right)^2 \right]
\end{aligned}$$

This of course can happen to be a counterexample for Theorem 3.2 depending on the first summand in the parenthesis. But $\Gamma^\star$ is in this case positive semidefinite, and therefore allows us to use Theorem 3.3.

Potts-Type spin Here, we have $\Gamma(s,t,s',t') = 1_{s=t=s'=t'}$. Therefore, the quadratic form yields:

$$\begin{aligned}
Q_\Gamma(f) &= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \big[\gamma(s,t) - \Gamma(s,t,s',t')\big] \\
&= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \big[\delta_{s,t} - \delta_{s,t,s',t'}\big] \\
&= \sum_s \left(\sum_{s'} f(s,s') \right)^2 - \sum_s f(s,s)^2
\end{aligned}$$

If $|\Sigma| = 2$, we have $\Gamma = \operatorname{diag}(1,0,0,1)$, and then, using (3.3), this is:

$$Q_\Gamma(f) = \frac{(a-c)^2}{2} - a^2 - c^2 = -\frac{(a+c)^2}{2}$$

Hence, we have superadditivity in this case. But once we increase $|\Sigma|$ to 3, the quadratic form is positive or negative, depending on the argument:

$$Q_\Gamma \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} = -3, \qquad Q_\Gamma \begin{pmatrix} 1/2 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & -1/2 \end{pmatrix} = 4.$$

Again, in this case $\Gamma^\star = \Gamma$, and hence by Theorem 3.3, we have superadditivity for all $\Sigma \neq \emptyset$.

Negative semidefinite Here we give an example where Theorem 3.2 gives more than Theorem3.3. Let $\Gamma(s,t,s',t') = 1_{s=s'\neq t=t'}$ that is:

$$\Gamma = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix} \qquad \Rightarrow \Gamma^\star = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$$

Observe that $\Gamma^\star$ is negative definite, hence Theorem 3.3 cannot be applied. But:

$$\begin{aligned} Q_\Gamma(f) &= \sum_s \sum_{s'} f(s,s') \sum_{t'} f(-s,t') - \sum_s f(s,s) f(-s,-s) \\ &= 2(a+b)(b+c) - 2ac = -\frac{(a+c)^2}{2} \end{aligned}$$

This is negative, therefore, using Theorem 3.2, we have superadditivity.

No superadditivity at all Let

$$\Gamma = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix} \qquad \Rightarrow \Gamma^\star = \begin{pmatrix} 1&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$$

Observe that $\Gamma^\star$ is indefinite, hence Theorem 3.3 cannot be used. Furthermore:

$$Q_\Gamma(f) = -\tfrac{1}{4} \cdot (a+c) \cdot (5a-c)$$

is indefinite either. Therefore, we have no hints on superadditivity in this case, because we are not given any simple bound on the error of superadditivity. We can be sure of it only in the case where we have some understanding on the Gibbs measure, which we have only for small enough β. But there we have a proof of the convergence of the free energy anyway, even with some estimate of the speed of convergence.

The Volunteer Model Consider the model given by:

$$\Gamma := \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

There we have:

$$\begin{aligned} Q_\Gamma(f) &= \sum_{s,t,s',t'} f(s,s')f(t,t') \cdot \big[\gamma(s,t) - \Gamma(s,t,s',t')\big] \\ &= \sum_{s',t'} f(1,s')f(1,t') - f(1,1)^2 = \left(\sum_{s'} f(1,s')\right)^2 - f(1,1)^2 \end{aligned}$$

Then, applying (3.3) for f:

$$Q_\Gamma(f) = \left(a - \tfrac{a+c}{2}\right)^2 - a^2 = \left(\tfrac{a-c}{2}\right)^2 - a^2 = -\frac{(a+c)(3a-c)}{4}$$

This is only in some part of the $[-1,1]^2$ negative, in others it is positive. Hence Theorem 3.2 cannot be applied at all. Note that in this case $\Gamma^\star = \Gamma$ is positive semidefinite. Still, by Theorem 3.3, we have superadditivity.

Toy Model Let $\Gamma(s,t,s',t') = 1_{s=t=s'=t'} - 1_{s=t\neq s'=t'}$:

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \Gamma^\star = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Of course $\Gamma^\star$ is indefinite, and again Theorem 3.3 fails to say anything about this. On the other hand:

$$\begin{aligned} Q_\Gamma(f) &= \sum_s \left(\sum_{s'} f(s,s')\right)^2 - \sum_s f(s,s)^2 + \sum_s f(s,-s)^2 \\ &= \frac{(a-c)^2}{2} - a^2 - c^2 + \frac{(a+c)^2}{2} = 0. \end{aligned}$$

This means that $Q(f)$ vanishes on the given subspace! Therefore, the Theorem gives additivity – up to the $o(N)$ error. Of course we have seen that for this model we can calculate the free energy directly, and observe the additivity in a trivial way.

Observe that this special case indeed shows that our Theorem 3.2 can be stronger than Theorem 3.3. And even if we combined the local partition function method with our trick of considering only a subspace for the convexity, in this case this would give the better result $-\frac{1}{2}(a-c)^2$.

Of course this model is, as we have seen, a very simple model. But there are also other Γ that exhibit additivity, as will be shown later in (8.1).

3.2. Intriguing Aspects of this System

The proofs in the next part were not our first attempts at proving anything. It is amazing how apt the proof of Talagrand in [15] circumvents the issues we had with other proofs. For the standard SK model, getting the convergence of the overlap is the most important step, after which a simple derivative by β is enough to compare the free energy to the formula.

In contrast for the multivariate SK model it is not even clear how the derivative of κ w.r.t. β should be computed, because the covariance of the Gaussian field in the exponent of the fixed point equation depends on κ, therefore this derivative is non-trivial.

The other strength of this model is the breadth of examples it can be applied to, Chapter 2 was of course only a null set. This quiver can be used to find interesting instances, e.g. in the last section we saw that the Theorem 3.2 can be used for Potts-type spins only if we have two states; for three states it fails at least without further ado.

There is one more thing: If the Parisi picture of the low temperature SK model has any interpretation as clusters of spins interacting due to their state and on multiple hierarchical levels, then this has to be a multivariate interaction, since it bundles many independent interactions into a single one. Our study of the compound model outlined in Section 2.3.4 was a first attempt to understand this connection.

Part II.

Further Generalizations and Proofs

4. The Model Revisited

In [18] we heuristically calculated the value of the free energy and some order parameters and showed that those order parameters have unique solutions for β small enough. Here, we give a rigorous proof for the conjecture on the thermodynamic limit of the free energy in the high temperature regime using the more standardizing methods presented in [15]. There, the famous smart path method – or Guerra interpolation – is brought to the next level, as there are used two of those: The first one replaces the usual β-derivative by interpolating between the full system (1.2) and a non interacting one. The second one decouples just a single spin creating a 'cavity' as in many parts of [16] to compute the overlap – a pivotal quantity.

In order for this to work, it is important to have an estimate of the overlap along every interpolation point in the first interpolation. Therefore, the idea in Talagrand's paper is to obtain estimates uniformly over a whole class of Hamiltonians in which both interpolations live. He thus introduces parameters a_{ij} that keep track of the fade out state of each interaction pair. This slightly generalizes the original SK model in the direction of the diluted SK model, but the full diluted SK model seems not to be amenable to this. At least this is not mentioned by Talagrand and we will see in Section 4.3 why this proof is too crude. This new method of Talagrand seems to be more powerful than others, at least we were not able to prove our statements using previous methodology.

Because of this generalization, we have to revisit most of our definitions. This is what we will do in the beginning of this chapter. Then, after introducing the class of Hamiltonians, we will restate the generalized and more exact versions of the theorems in Section 1.3. After that, we will investigate what models can be handled by the proofs we have here and show the self-averaging property of the free energy.

Chapter 5 will cover the proofs of the theorems. It starts by stating the main lemma and giving its proof. Then, Theorems 4.3 and 4.4 are derived from this. Since in our model (1.2) the interaction term $g_{ij}(\sigma_i, \sigma_j)$ can be much more complicated, the calculations in [15] carry over in principle, but are somewhat more intricate. Finally, we give the first proofs on the asymptotic independence.

Chapter 6 then gives the proofs on the TAP-equations. Interestingly enough for that proof we again capitalize on the concept of a class of Hamiltonians. This enables us to stay at fixed β whereas one usually has to cope with a $\beta^- := \sqrt{\frac{N}{N-1}}\beta$.

Finally, in the end of this thesis, we will investigate how these methods can be generalized for more general Gaussian random fields on infinite Σ and will give the heuristics that lead to the Parisi formula.

4.1. Definitions of the SK Model with Dilutions

In the line of [15], we start by considering the slightly generalized Hamiltonian:

$$H_{\mathbf{A}}(\boldsymbol{\sigma}) := \sum_{1 \le i < j \le N} \sqrt{a_{ij}}\, g_{ij}(\sigma_i, \sigma_j), \tag{4.1}$$

where $\mathbf{A} = (a_{ij})_{ij}$ is a symmetric matrix with entries $a_{ij} \ge 0$ and $a_{ii} = 0$ for all i. We will call this a **dilution matrix**. The standard case is $a_{ij} = \frac{\beta^2}{N}$ for all $i \ne j$.

We adapted the term dilution matrix from the so-called Diluted SK model (cf. [16, Chapter 6]), where $\mathbf{A}$ is a random matrix with Bernoulli variable entries. This corresponds to a Bernoulli percolation on the complete graph – one variant of the Erdős-Rényi random graph.

Depending on the structure of $\mathbf{A}$, different regimes will occur. In many interesting cases $\mathbf{A}$ does not distinguish the sites. We formalize this notion in the following

Definition 4.1. *Let* $\mathbf{A}$ *be a dilution matrix. Then* $\mathbf{A}$ *is said to have* ***symmetry between sites*** *if for all* $1 \le i, j \le N$, $i \ne j$, *there exists a permutation* $f_{ij}\colon \{1, \dots, N\} \to \{1, \dots, N\}$ *with the following three properties:*

$$f_{ij}(i) = j, \qquad f_{ij}(j) = i, \qquad a_{f_{ij}(k), f_{ij}(l)} = a_{k,l} \qquad \forall 1 \le k, l \le N.$$

Order parameters for the Dilution First, we have to revisit the self-consistency equations for the order parameter because they will be a central ingredient in the definition of the model. Of course in this setting, depending on $\mathbf{A}$, $\mathbb{E}\mu_i(s)\mu_i(s')$ does not have to be the same for all $i \leq N$. Hence, we now consider $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_N)$, a sequence of symmetric matrices $\big(\kappa_i(s,s')\big)_{s,s' \in \Sigma}$ with $\kappa_i(s,s') \in [0,1]$ and $\sum_{s,s'} \kappa_i(s,s') = 1$. As before, we set $\pi_i(s) := \sum_{s' \in \Sigma} \kappa_i(s,s')$, $s \in \Sigma$, and assume further that $0 \preceq \kappa_i$ for all i.

Now, as we saw heuristically in Section 1.2.3, the local field of σ_N is split into a part $Y_N(\cdot)$ that depends on the disorder and $\Phi_\kappa(\cdot)$ that captures the fluctuation due to the Gibbs measure. Recall from (1.10) and (1.11) that both are basically given by looking at

$$\sum_{s,s'} \Gamma(s,t,s',t') \cdot \frac{1}{N} \sum_{j<N} \mu_j^{(N-1)}(s) \mu_j^{(N-1)}(s'), \qquad t, t' \in \Sigma.$$

This is approximated by $\Gamma_\kappa(t,t') = \sum_{s,s'} \Gamma(s,t,s',t')\kappa(s,s')$. Now looking back at the original heuristical calculation one sees that the correct quantity here is for any $i \leq N$:

$$\begin{aligned} \Gamma_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(t,t') &:= \sum_{j=1}^{N} a_{ij} \Gamma_{\kappa_j}(t,t') \\ &\approx \sum_{s,s'} \Gamma(s,t,s',t') \cdot \sum_{j=1}^{N} a_{ij} \mu_j^{(N-1)}(s) \mu_j^{(N-1)}(s') \end{aligned}$$

and similarly:

$$\Phi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s) := \sum_{j=1}^{N} a_{ij} \Phi_{\kappa_j}(s).$$

Remark that in both previous displays the summand for $j = i$ is zero since $a_{ii} = 0$. We will use this tacitly all the time from now on.

Since we assume $0 \preceq \kappa_i$ for all i, both Γ_{κ_i} and $\Gamma_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}$ are positive semidefinite for all $i \leq N$. Therefore, there is an i.i.d. sequence of Gaussian fields $\big(Y_i(s)\big)_s$ with covariance matrix $\Gamma_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}$, $i \leq N$. Then, the fixed point equations are given using the functions

$$\Pi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s) := \frac{1}{Z_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}} p(s) \exp\left[Y_i(s) + \Phi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s)\right], \tag{4.2}$$

$$Z_{\boldsymbol{\kappa}}^{(\mathbf{A},i)} := \sum_{s\in\Sigma} p(s) e^{Y_i(s)+\Phi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s)}, \tag{4.3}$$

Therefore, we get again:

Lemma 4.2. *(a) Given* $\mathbf{A}$*,* Γ *and* p*, the set of equations*

$$\kappa_i(s,s') = \mathbb{E}_Y \Pi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s)\Pi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s'), \qquad \forall s,s' \in \Sigma,\ \forall i \leq N \tag{4.4}$$

does have a fixed point solution. For any solution $\boldsymbol{\kappa}$*, we have* $0 \preceq \kappa_i$ *for all* $i \leq N$*.*

(b) There is a $L < \infty$*, only depending on* Γ*, s.t. a unique solution exists if the matrix norm* $\|\mathbf{A}\|_1 < \frac{1}{L}$ *is small enough.*

Part *(a)* is again a consequence of Brouwer's fixed-point theorem. Part *(b)* is proved by the Banach fixed-point theorem – the appropriate Lipschitz constant for the map $\boldsymbol{\kappa} \mapsto \left(\mathbb{E}\Pi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(\cdot)\Pi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(\cdot)\right)_{i\leq N}$ will be estimated in section 5.4.

Even without *(b)*, all our error bounds will hold, if we pick and fix any of the solutions that are already guaranteed to exist. Obviously, if we have convergence, then the limits we will calculate cannot depend on the choice we made by picking one of the solutions, and hence are independent of that choice.

Fixed point equation for the augmented Hamiltonian Given $\mathbf{A}$, Γ, and p, we now fix any/the solution $\boldsymbol{\kappa}$ of the corresponding fixed point equation. Then, also the laws of the Y_i are fixed. We can rewrite the exponent in (4.2) as:

$$Y_i(s) + \Phi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s) \overset{\mathcal{L}}{=} \sum_{j=1}^{N} \sqrt{a_{ij}}\, g_i^{(j)}(s) + \sum_{j=1, j\neq i}^{N} a_{ij}\Phi_{\kappa_j}(s),$$

where the $g_i^{(j)}$ is a Gaussian field with covariance matrix Γ_{κ_j}, independent of the $Y_i(\cdot)$. Hence, if we augment $H_{\mathbf{A}}$ by a local field at all sites with this specific law:

$$H_{\mathbf{A},\mathbf{B}}(\boldsymbol{\sigma}) = H_{\mathbf{A}}(\boldsymbol{\sigma}) + \sum_{i,j} \sqrt{b_{ij}} g_i^{(j)}(\sigma_i) + \sum_{i,j} b_{ij} \Phi_{\kappa_j}(\sigma_i),$$

where $\mathbf{B} = (b_{ij})_{i \le N}$ is another dilution matrix, then the total local field at σ_i becomes:

$$\begin{aligned} Y_i(s) + \Phi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s) + \sum_j \sqrt{b_{ij}} g_i^{(j)}(s) + \sum_j b_{ij} \Phi_{\kappa_j}(s) \\ \overset{\mathcal{L}}{=} \sum_j \sqrt{a_{ij} + b_{ij}} g_i^{(j)}(s) + \sum_j (a_{ij} + b_{ij}) \Phi_{\kappa_j}(s) \end{aligned}$$

Hence, the corresponding fixed point equation is given using $\Pi_{\boldsymbol{\kappa}}^{(\mathbf{A}+\mathbf{B},\bullet)}$.

Conversely, if $\mathbf{A} + \mathbf{B} = \mathbf{A}' + \mathbf{B}'$, then both systems share the same solution of the fixed point equation. Therefore, as long as the sum $\mathbf{A} + \mathbf{B}$ stays constant, $\boldsymbol{\kappa}$ stays constant, and hence the covariances Γ_{κ_i} stay constant, which allows us to reuse the $g_i^{(j)}$. This is the crucial point behind the following definition of a class of Hamiltonians.

On the other hand, all the calculations we will perform would in principle carry through for any fixed Gaussian field $g_i^{(j)}(\cdot)$ with covariance matrix say $\Gamma_{\hat{\kappa}_j}(\cdot)$. However, for the final interpolation point at $x = 0$, we need to have that the Hamiltonian at that point is precisely the exponent of the $\Pi(\cdot)$ in the fixed point equation, so that terms of the form $\hat{\kappa}_j(t,t') - \mathbb{E}\Pi_j(t)\Pi_j(t')$ vanish. Hence we need the class of Hamiltonians to contain $H_{\mathbf{A}}$ as well as those Hamiltonians with independent spins, but correct local fields.

The Model Given Γ, p and a dilution matrix $\mathbf{C}$, we fix the corresponding solutions $\boldsymbol{\kappa}$ for the fixed point equations. Let g_{ij} $(1 \le i < j \le N)$ be i.i.d. copies of a Gaussian field with covariance matrix Γ, and let further $g_i^{(j)}$ $(1 \le i, j \le N,\ i \ne j)$ be i.i.d. copies of Gaussian fields with covariance matrices Γ_{κ_j}. We denote by $\mathbb{E}$ the expectation w.r.t. g_{ij} and $g_i^{(j)}$. Given the dilution matrices $\mathbf{A} = (a_{ij})_{i \ne j}$ and $\mathbf{B} = (b_{ij})_{i \ne j}$ with $\mathbf{A}+\mathbf{B} = \mathbf{C}$, we define for $\boldsymbol{\sigma} \in \Sigma^N$ the Hamiltonian

$$H_{\mathbf{A},\mathbf{B}}(\boldsymbol{\sigma}) = \sum_{i<j} \sqrt{a_{ij}} g_{ij}(\sigma_i, \sigma_j) + \sum_{i,j} \sqrt{b_{ij}} g_i^{(j)}(\sigma_i) + \sum_i \Phi_{\boldsymbol{\kappa}}^{(\mathbf{B},i)}(\sigma_i), \qquad (4.5)$$

Then, $\mathcal{H}$ is the following class of Hamiltonians:

$$\mathcal{H} = \mathcal{H}_N(\mathbf{C}, \Gamma, p) := \left\{ H_{\mathbf{A},\mathbf{B}}(\boldsymbol{\sigma}) \mid a_{ij} + b_{ij} = c_{ij}\ \forall i, j \leq N \right\}. \tag{4.6}$$

Now for all $H \in \mathcal{H}$, we denote as before the Gibbs probability by $P_H(\cdot)$, the Gibbs expectation on any number of replicas by $\langle \cdot \rangle_H$, and the averaged expectation by $\nu_H(\cdot)$.

Further Definitions In the next section we will state convergence rates for several important objects. They will be consequences of recursive applications of the Main Lemma 5.1 and hence give rise to the following definitions, where L_0, cf. (5.7), is a constant depending only on Γ:

$$\mathbf{M} := (m_{ij})_{ij} = \sum_{k=0}^{\infty} (L_0 \mathbf{C})^k, \quad \mathbf{b} := \left(\|c_{\bullet i}\|_2 \right)_{i \leq N} = \left(\sqrt{\sum_{j=1}^{N} c_{ji}^2} \right)_{i \leq N}, \quad \text{and} \tag{4.7}$$

$$\mathbf{w} := \mathbf{M}\mathbf{b}. \tag{4.8}$$

Of course $\mathbf{M}$ is only finite if all eigenvalues of $L_0\mathbf{C}$ have absolute value less than 1. This is given if $\mathbf{C}$ is s.t. $\|\mathbf{C}\|_2 < \frac{1}{L_0}$. The results below are hence only useful at high enough temperature.

Interestingly enough, if $\mathbf{C}$ has **symmetry between sites**, then, given that $\mathbf{M}$ exists, $\mathbf{w}$ is a constant vector as we will see in Section 4.3. For instance, in the standard case $c_{ij} = \frac{\beta^2}{N}$, it is given asymptotically as:

$$\mathbf{w} \sim \frac{\beta^2}{1 - L_0 \beta^2} \cdot \left(\tfrac{1}{\sqrt{N}}, \ldots, \tfrac{1}{\sqrt{N}} \right) \tag{4.9}$$

as soon as $\beta < \frac{1}{\sqrt{L_0}}$. That is for all $\beta < \beta_0 := \frac{1}{4\sqrt{K}}$.

Comparison with Talagrand's Scheme Our layout of this scheme is slightly different than Talagrand's original one in [15]. This is due to the fact that in the standard SK model the covariance matrices are obviously only real constants. Hence we have $g_i^{(j)} = \sqrt{q_j} z_i^{(j)}$, where $z_i^{(j)}$ are i.i.d. standard Gaussians. Then, $\sum_j \sqrt{b_{ij}} g_i^{(j)}$ is a centered Gaussian with variance $\sum_j b_{ij} q_j$, hence this can be simplified in that case by use of a vector setting instead of the matrix $\mathbf{B}$.

4.2. Results

Now fix Γ, p, and $\mathbf{C}$, that is fix a class $\mathcal{H}$. Then, we have for the free energy:

Theorem 4.3. *For every $H \in \mathcal{H}$:*

$$\left|\mathbb{E}\log\sum_{\boldsymbol{\sigma}}\exp\big(H(\boldsymbol{\sigma})\big) - p_N\right| \le L \cdot \sum_i w_i^2,$$

where

$$p_N := -\tfrac{1}{4}\sum_{i,j} c_{ij}\Big[\gamma(\pi_i,\pi_j) - \Gamma^\star(\kappa_i;\kappa_j)\Big] + \sum_{i\le N}\mathbb{E}\log\mathrm{Tr}_s\, e^{Y_i(s)+\Phi_{\boldsymbol{\kappa}}^{(\mathbf{C},i)}(s)}. \tag{4.10}$$

As both terms on the left hand side are approaching ∞ as $N\to\infty$, this result is useful if the upper bound is finite, i.e. $\|\mathbf{w}\|_2$ has to be bounded. If $c_{ij} = \frac{\beta^2}{N}$, $i\neq j$, this holds due to (4.9). For other examples see Section 4.3.

As our most important result for the high temperature behavior, we obtain convergence of the 'overlap':

Theorem 4.4. *For any given numbers $\alpha_1,\dots,\alpha_N \ge 0$ and any $\boldsymbol{\eta},\boldsymbol{\eta}' \in \Sigma^N$, we have:*

$$\nu_H\left[\Big(\sum_{i\le N}\alpha_i\big(\delta_i(\eta_i) - \pi_i(\eta_i)\big)\Big)^2\right] \le 2\sum_i \alpha_i^2 + 2\Big(\sum_i \alpha_i w_i\Big)^2 \quad \textit{and}$$

$$\nu_H\left[\Big(\sum_{i\le N}\alpha_i\big(\delta_i^{1,2}(\eta_i,\eta_i') - \kappa_i(\eta_i,\eta_i')\big)\Big)^2\right] \le 2\sum_i \alpha_i^2 + 2\Big(\sum_i \alpha_i w_i\Big)^2$$

uniformly over all $H\in\mathcal{H}$.

Here, the first terms on the right hand sides converge for $\alpha_i \equiv N^{-a}$, as long as $a > \frac{1}{2}$. The interesting case is $\alpha_i \equiv \frac{1}{N}$. Then, convergence means that $\|\mathbf{w}\|_1$ has to be bounded, which is again true in the case (4.9).

We now turn our attention to two consequences of Theorem 4.4 regarding the high temperature regime. They will be proved in Section 5.3

Proposition 4.5. *Let $\alpha_1, \dots, \alpha_N \geq 0$. Fix any $\boldsymbol{\eta} \in \Sigma^N$. Then, we have for any $H \in \mathcal{H}$:*

$$\sum_{i<j} \alpha_i \alpha_j \mathbb{E}\Big[\big\langle \, \big(\delta_i(\eta_i) - \mu_i(\eta_i)\big) \cdot \big(\delta_j(\eta_j) - \mu_j(\eta_j)\big) \, \big\rangle_{\mathcal{H}}^2 \Big] = L \cdot O_2(\boldsymbol{\alpha}),$$

where by $O_2(\boldsymbol{\alpha})$ we denote the right hand side of Theorem 4.4.

If $\mathbf{C}$ is constant off the diagonal, obviously, for $\alpha_i \equiv \frac{1}{N}$, this signifies decay of correlations for say σ_1 and σ_2. Therefore, in that case we will see:

Theorem 4.6. *Let $\mathbf{C}$ be constant everywhere but on the diagonal. Fix $H \in \mathcal{H}$ and $n \in \mathbb{N}$. Let $P_{1 \cdots n}(s_1, \dots, s_n)$ be the marginal distribution of the first n spins under the Gibbs distribution $\langle \cdot \rangle_H$. Let $\boldsymbol{\mu}_n := \mu_1 \otimes \cdots \otimes \mu_n$ be the independent product of the first n marginals. Then, we have*

$$\mathbb{E}\big[\big(|P_{1\cdots n} - \boldsymbol{\mu}_n|_{\mathrm{TV}}\big)^2\big] \leq \frac{L}{N} + \frac{L}{N^2}\left(\sum_i w_i\right)^2,$$

where $|\cdot|_{\mathrm{TV}}$ is the total variation and L depends on Γ and n only.

The last result is a generalization of Theorem 1.4.15 in [16]. We will see more results on this asymptotic independence in Chapter 6, namely a kind of central limit theorem and the TAP equations.

4.3. Conditions for Convergence

We now investigate under which circumstances the previous results are meaningful. We first give two methods for evaluating whether there is convergence. Then, we look at several examples of $\mathbf{C}$ and discuss them. This is all done independently of the structure of the spins and their pair interactions. Therefore, this will hold for all (Σ, Γ, p). Only by using L_0 as defined in (5.7), we will depend on Γ.

First, we rewrite the two main theorems for the most common cases. For every $H \in \mathcal{H}$ we have the free energy formula:

$$\tfrac{1}{N}\left|\mathbb{E} \log \sum_{\boldsymbol{\sigma}} \exp\big(H(\boldsymbol{\sigma})\big) - p_N\right| \leq \tfrac{L}{N} \cdot \sum_i w_i^2 = \tfrac{L}{N} \cdot \|\mathbf{w}\|_2^2 =: O_1,$$

As $\frac{1}{N} p_N$ is of order 1, this result is useful if the upper bound is $o(1)$. Furthermore:

$$\nu_H\left[\left(\frac{1}{N}\sum_{i\le N}\left(\delta_i^{1,2}(s,s')-\kappa_i(s,s')\right)\right)^2\right] \le \tfrac{2}{N}+\tfrac{2}{N^2}\left(\sum_i w_i\right)^2$$
$$= \tfrac{2}{N}+2\left(\tfrac{\|\mathbf{w}\|_1}{N}\right)^2 =: O_2.$$

Hence, in order to put things together, they are, respectively:

$$O_1 = o(1) \iff \|\mathbf{w}\|_2^2 = o(N), \qquad O_2 = o(1) \iff \|\mathbf{w}\|_1 = o(N).$$

Under what circumstances is this useful? First of all, $\mathbf{M}$ has to exist. This is of course equivalent to the spectral radius of $L_0\mathbf{C}$ being less than 1 or

$$\rho = \|\mathbf{C}\|_2 < \frac{1}{L_0}.$$

We will call this the high enough temperature constraint. But in order for convergence in Theorems 4.3 and 4.4, we have to investigate further conditions. We now give two kinds of conditions. First, most often, $\mathbf{w}$ can be calculated, as we already stated. Second, in other circumstances, there are more general conditions, as the ones in Talagrand's original article [15]. Last, for the sake of completeness, we show a slight generalization of Talagrand's conditions when considering the free energy convergence.

Direct computation of w In many interesting cases it is possible to compute $\mathbf{w}$ directly. Indeed, if all eigenvalues of $L_0\mathbf{C}$ are less than 1, then we have:

$$\mathbf{M} = \sum_{k=0}^{\infty}(L_0\mathbf{C})^k = (I-L_0\mathbf{C})^{-1}$$

and hence if this inverse exists (i.e. $\mathbf{C}$ has $\rho < \frac{1}{L_0}$)

$$\mathbf{w} = \mathbf{M}\mathbf{b} \iff \mathbf{b} = (I-L_0\mathbf{C})\,\mathbf{w} \tag{4.11}$$

This solution can be calculated for several cases.

Assume that all rows of $\mathbf{C}$ are permutations of the same vector. This is of course the case when there is symmetry between sites, and therefore for the standard $c_{ij} = \frac{\beta^2}{N}$ as we will see in the examples.

Under this assumption, $\mathbf{b} = b \cdot (1, \ldots, 1)$ is a constant vector and, because $(1, \ldots, 1)$ is an eigenvector of $\mathbf{C}$, the above system of equations is solved by the constant vector $\mathbf{w} = w \cdot (1, \ldots, 1)$ where

$$w = \frac{b}{1 - L_0 \sum_i c_{1i}}. \tag{4.12}$$

Then:

$$\|\mathbf{w}\|_2^2 = N \cdot w^2 \qquad \text{and} \qquad \|\mathbf{w}\|_1 = N \cdot w. \tag{4.13}$$

Talagrand's Conditions In his article, Talagrand assumes the following conditions (c.f. [15, (1.8)–(1.10)]):

$$\sum_{j=1}^{N} c_{ij} \leq a, \qquad \forall 1 \leq i \leq N \tag{4.14}$$

$$c_{ij} \leq b_N, \qquad \forall 1 \leq i, j \leq N. \tag{4.15}$$

The first constraint focuses on high temperature and the second one on finite connectivity, as Talagrand puts it. Given $\mathbf{C}$, the optimal a is the matrix norm $\|C\|_1 = \max_i \sum_j c_{ij}$, the optimal b_N is $\max_{ij} c_{ij}$. Then, under the further assumption of high enough temperature, $a \leq \frac{1}{2L_0}$, he calculates the convergence rates:

$$O_1 \leq L \cdot ab_N \leq b_N, \qquad O_2 \leq 2L \cdot ab_N \leq 2b_N$$

Norm Conditions for the free energy formula Observe that, since $\|\cdot\|_2$ is a sub-additive matrix norm and because $\|\mathbf{C}\|_2 < \frac{1}{L_0}$, elementary calculations yield:

$$\|\mathbf{w}\|_2^2 \leq \|\mathbf{M}\|_2^2 \cdot \|\mathbf{b}\|_2^2 \leq \left(\sum_{k=1}^{\infty} \|L_0 \mathbf{C}\|_2^k \right)^2 \cdot \|\mathbf{b}\|_2^2 = \frac{1}{(1 - L_0 \|\mathbf{C}\|_2)^2} \cdot \|\mathbf{C}\|_F^2, \tag{4.16}$$

where $\|\mathbf{C}\|_F = \sqrt{\sum_{i,j} c_{ij}^2}$ is the Frobenius or Hilbert-Schmidt norm. Therefore, for the free energy, this reduces to the condition

$$\|\mathbf{C}\|_F = o(1).$$

Recall that $\|\cdot\|_F \le \sqrt{k}\|\cdot\|_2 \le \sqrt{N}\|\cdot\|_2$, where k is the rank of the matrix. For the L_N convergence this is no improvement on Talagrand's bounds above.

4.3.1. Examples on Graphs

The most natural examples are those where all the non-vanishing entries of the dilution matrix $\mathbf{C}$ are the same value c_N, which reduces the structure to graphs. Hence, consider a simple graph G_N with N vertices and denote its set of edges by E_N. Let $\mathbf{A}$ be the adjacency matrix and $\mathbf{d}$ the vector of degrees, that is $\mathbf{d} = (1, \ldots, 1)\mathbf{A}$, and denote the maximal degree by $\Delta_N := \|\mathbf{d}\|_\infty$. Then we set the dilution matrix

$$\mathbf{C} := c_N \cdot \mathbf{A},$$

where $c_N > 0$ is some constant depending on the number of vertices N.

First, we calculate some of the matrix norms:

$$\|\mathbf{C}\|_1 = \max_i \sum_j c_{ij} = c_N \Delta_N, \qquad \|\mathbf{C}\|_F^2 = c_N^2 \sum_{ij} a_{ij} = 2c_N^2 \cdot |E_N|.$$

The spectral radius $\|\mathbf{A}\|_2$ of a graph is a well investigated quantity that heavily depends on the structure of the graph. Therefore, we will only investigate it in the examples. Generally, we can simply bound it by $\|\mathbf{A}\|_1$ which yields the bound of Talagrand. Now, for $\mathbf{b}$ we have:

$$b_i = \sqrt{\sum_{j=1}^N c_{ij}^2} = c_N \cdot \sqrt{d_i}, \qquad \|\mathbf{b}\|_1 = c_N \cdot \sum_i \sqrt{d_i},$$
$$\|\mathbf{b}\|_2^2 = c_N^2 \cdot \sum_i d_i = c_N^2 \cdot |E_N|.$$

The direct approach (4.11) gives something similar to Laplace's equation on the graph:

$$c_N \sqrt{d_i} = w_i - L_0 c_N \sum_{j \sim i} w_j \qquad \forall i \le N$$

Talagrand's conditions assume that $c_N \Delta_N < \frac{1}{2L_0}$ and infer $b = c_N^2 \Delta_N = O(c_N)$ as the rates of convergence.

Now, we show some examples of graphs and investigate the convergence on them.

Regular graphs For a d-regular graph (i.e. $d_i = d$ constant for all i), it is known that the spectral radius $\rho(G_N) = d$. Hence $\mathbf{M}$ exists, if $c_N < \frac{1}{L_0 d}$. Furthermore, we can calculate $\mathbf{w}$ directly. Indeed, here $\mathbf{A}$ has exactly d ones on every row and (4.12) gives:

$$w = \frac{c_N \sqrt{d}}{1 - L_0 c_N d}, \qquad \|\mathbf{w}\|_2^2 = w^2 N = O(N c_N^2 d) = O(N c_N), \tag{4.17}$$
$$\|\mathbf{w}\|_1 = wN = O(N c_N \sqrt{d}) = O(N\sqrt{c_N}).$$

For convergence, this has to be $o(N)$ which implies $c_N = o(1)$.

If we compare this with (4.16), we have $\|\mathbf{C}\|_2 = c_N \cdot \rho(A) = c_N\, d$ which has to be bounded away from $1/L_0$ and again gives the constraint on β. For the second term, we have:

$$\|\mathbf{C}\|_F^2 = c_N^2 \sum_{ij} a_{ij} = c_N^2 \cdot d\, N \le c_N^2\, N\, \Delta_N \le O(c_N\, N)$$

which, again, has to be $o(N)$, and hence $c_N = o(1)$.

Complete Graph This is the case we have in the standard SK model:

$$c_{ij} = \frac{\beta^2}{N}, \qquad i \neq j$$

Because the complete graph is regular with degree $N-1$, the spectral radius is $N-1$. Hence, we have convergence if $\beta^2 < \beta_0 := \frac{1}{L_0}$. Of course we also have:

$$\|\mathbf{C}\|_1 = \beta^2 \frac{N-1}{N}$$

Therefore, this is the same in this case.

For the convergence rates we obtain:

$$O_1 = L \cdot \|\mathbf{w}\|_2^2 = O(1), \qquad O_2 = L \cdot \|\mathbf{w}\|_1 = O(\sqrt{N})$$

Bounded Degree Assume that there is a fixed $\Delta \in \mathbb{N}$, s.t. all the graphs in the sequence have maximal degree less or equal than Δ. Then, it is known and easy to see that if the all the graphs in the sequence have a uniformly bounded degree Δ – e.g. a sequence of finite subgraphs of $\mathbb{Z}^d$ as in the Edwards-Anderson model –, then the spectral radii are bound by this Δ. But since – at least if we assume that we have not a vast majority of not connected points – the number of edges should be at least the number of vertices, this implies that for convergence we need to have $c_N = o(1)$, because $|E_N| \leq \Delta N$ and this is the only necessary condition in bounded degree graphs.

Now, in the diluted SK model, all vertices have a bounded expected degree. Therefore, these calculations are too crude to handle it.

The Star An interesting case with varying degrees is the $N-1$ star which is a tree where all non-root vertices are attached to the root only. Then, all non-root vertices have degree 1 and the root has degree $N-1$. We set $c_N = \frac{\beta^2}{\sqrt{N}}$. For $\mathbf{w}$ we have the following system of equations:

$$\begin{aligned} c_N\sqrt{N-1} &= w_1 - L_0 c_N \sum_{j=2}^{N} w_j, \\ c_N &= w_j - L_0 c_N w_1, \qquad \forall j > 1. \end{aligned}$$

This implies, setting $h := c_N\sqrt{N-1}$:

$$\begin{aligned} w_1 &= \frac{h + L_0 c_N^2 (N-1)}{1-(L_0 c_N)^2(N-1)} = \frac{h + L_0 h^2}{1-(L_0 h)^2}, \\ w_j &= c_N(1+L_0 w_1), \qquad j > 1 \\ \|\mathbf{w}\|_1 &= w_1 + (N-1)\cdot c_n \cdot (1+L_0 w_1), \\ \|\mathbf{w}\|_2^2 &= w_1^2 + (N-1)\cdot c_N^2 \cdot (1+L_0 w_1)^2 \end{aligned}$$

Since the spectral radius of the star is readily seen to be $\sqrt{N-1}$, for the existence of $\mathbf{M}$ we first of all need $h < \frac{1}{L_0}$, which means $c_N = \frac{\beta^2}{\sqrt{N}}$ with $\beta < \frac{1}{\sqrt{L_0}}$. Then, we also have $w_1 \leq Lh$ which gives us:

$$O_1 \leq Lh + L\sqrt{N}(1+Lh) = O(\sqrt{N}), \quad O_2 \leq (Lh)^2 + \beta^2(1+Lh)^2 = O(1).$$

Observe that $\|\mathbf{C}\|_1 = c_N(N-1) \leq \beta^2\sqrt{N}$ is not bounded and in Talagrand's conditions this would not even be high temperature.

4.3.2. Non-constant Examples

Hierarchical Example We can divide the N spins into M subsets and let the interaction be c_1/N inside a block and c_0/N in between two blocks. If we set $N_0 = N$ and $N_1 = N_0/M$, then this has the following form (here of course for $M = 3$):

$$\mathbf{C} = \left(\begin{array}{c:c:c} \boxed{\begin{matrix}0 & c_1 & c_1\\ c_1 & 0 & c_1\\ c_1 & c_1 & 0\end{matrix}} & c_0 & c_0 \\ \hdashline c_0 & \boxed{\begin{matrix}0 & c_1 & c_1\\ c_1 & 0 & c_1\\ c_1 & c_1 & 0\end{matrix}} & c_0 \\ \hdashline c_0 & c_0 & \boxed{\begin{matrix}0 & c_1 & c_1\\ c_1 & 0 & c_1\\ c_1 & c_1 & 0\end{matrix}} \end{array}\right)$$

First, we look at the condition $\|\mathbf{C}\|_2 \leq \frac{1}{L_0}$. Therefore, we begin by bounding:

$$\|\mathbf{C}\|_2 \leq \|\mathbf{C}\|_1 = c_0 N_0 + (c_1 - c_0) N_1$$

Hence, if $c_0 = \frac{\beta_0^2}{N}$ and $c_1 = \frac{\beta_1^2}{N}$, we get the condition:

$$\beta_0^2 + \frac{\beta_1^2 - \beta_0^2}{M} < \frac{1}{L_0}$$

Now, the interesting case is $\beta_0 \neq \beta_1$. First, if $\beta_0^2 < \frac{1}{L_0}$, our condition reduces to:

$$\beta_1^2 < \beta_0^2 + M(\tfrac{1}{L_0} - \beta_0^2)$$

Hence, if we have say $\beta_0^2 = \frac{1}{2L_0}$, then $\mathbf{M}$ exists for all $\beta_1^2 < \frac{1}{2L_0}(1 + M)$. This means that if we divide the system into $M = 2$ blocks, we can overcompensate. That is, inside the block we have $\frac{1}{L_0} < \beta_1^2 < \frac{3}{2}\frac{1}{L_0}$. And we can obtain any β_1 if M is big enough!

On the other hand, if inside the blocks the spins are less interacting than outside and $\beta_0^2 > \frac{1}{L_0}$, then:

$$M(\beta_0^2 - \tfrac{1}{L_0}) < \beta_0^2 - \beta_1^2$$

This is only possible if $\beta_1^2 < \frac{1}{L_0}$. Here, a very interesting case is $\beta_1 = 0$, that is, inside the boxes there is no direct interaction, only with the other boxes there is a direct interaction. This implies $\beta_0^2 < \frac{1}{L_0} \cdot (1 + \frac{1}{M-1})$, which means for $M = 2$, we can still have double the critical β in the outside interaction.

Of course we have symmetry of spins and can calculate $\mathbf{w}$ by (4.12):

$$w = \frac{\sqrt{c_0^2 N_0 + (c_1^2 - c_0^2) N_1}}{1 - L_0\big(c_0 N_0 + (c_1 - c_0) N_1\big)} = \frac{L}{\sqrt{N}} \sqrt{\beta_0^4 + \frac{\beta_1^4 - \beta_0^4}{M}}$$

This, again, yields $O_1 = O(1)$ and $O_2 = O(\sqrt{N})$. Hence in the above cases, we indeed have convergence.

Two Sets of Spins In the example with the star we saw that interesting things can happen when symmetry of spins fails. Now, we dive into a generalization of that.

Assume three numbers $a_N, b_N, c_N > 0$ and consider the matrix of the form:

$$\mathbf{C} = \left(\begin{array}{ccc:ccccc}
0 & a_N & a_N & & & & & \\
a_N & 0 & a_N & & & c_N & & \\
a_N & a_N & 0 & & & & & \\
\hdashline
 & & & 0 & b_N & b_N & b_N & b_N \\
 & & & b_N & 0 & b_N & b_N & b_N \\
 & c_N & & b_N & b_N & 0 & b_N & b_N \\
 & & & b_N & b_N & b_N & 0 & b_N \\
 & & & b_N & b_N & b_N & b_N & 0
\end{array}\right)$$

That is, the first M spins have interaction a_N with each other, the last $N - M$ spins have interaction b_N and the interaction between spins of the first and second group are c_N.

Then, the star was the example with $M = 1$, $a_N = b_N = 0$ and $c_N = \beta^2/\sqrt{N}$. Another interesting example is the macroscopic pair: $M = 2$, $a_N = \beta^2$, $b_N = c_N = \beta^2/N$.

As with the star, we can assume that $\mathbf{w}$ is constantly w on the first M indices and constantly w' on the last $N-M$. This gives the system of equations:

$$\begin{aligned}\sqrt{\bar{M}a_N^2+\bar{N}c_N^2} &= w - L_0 a_N \bar{M} w - L_0 c_N \bar{N} w',\\ \sqrt{\bar{M}c_N^2+\bar{N}b_N^2} &= w' - L_0 c_N \bar{M} w - L_0 b_N \bar{N} w'\end{aligned}$$

with $\bar{M} := M-1$ and $\bar{N} := N-M+1$. By Cramer's rule, this gives the solution:

$$\begin{aligned}w &= \tfrac{1}{D}\cdot\big(\sqrt{\bar{M}a_N^2+\bar{N}c_N^2}\cdot(1-L_0 b_N\bar{N})+\sqrt{\bar{M}c_N^2+\bar{N}b_N^2}\cdot L_0 c_N\bar{N})\big),\\ w' &= \tfrac{1}{D}\cdot\big(\sqrt{\bar{M}c_N^2+\bar{N}b_N^2}\cdot(1-L_0 a_N\bar{M})+\sqrt{\bar{M}a_N^2+\bar{N}c_N^2}\cdot L_0 c_N\bar{M})\big)\\ D &= (1-L_0 a_N\bar{M})\cdot(1-L_0 b_N\bar{N})+(L_0 c_N)^2\bar{M}\bar{N}\end{aligned}$$

if $D\neq 0$, that is, when $a_N<\frac{1}{L_0 M}$ and $b_N<\frac{1}{L_0(N-M+1)}$, which gives the high enough temperature condition. Then, we have:

$$\begin{aligned}\|\mathbf{w}\|_1 &\leq L\sqrt{\tfrac{1}{M}+Nc_N^2}\cdot M\,(1+Nc_N)+L\sqrt{\tfrac{1}{N}+Mc_N^2}\cdot N\,(1+Nc_N)\\ &= L\big(M\sqrt{\tfrac{1}{M}+Nc_N^2}+N\sqrt{\tfrac{1}{N}+Mc_N^2}\big)\cdot(1+Nc_N),\\ \|\mathbf{w}\|_2^2 &\leq L(\tfrac{1}{M}+Nc_N^2)\cdot M\,(1+N^2c_N^2)+L(\tfrac{1}{N}+Mc_N^2)\cdot N\,(1+N^2c_N^2)\\ &= 2L\big(1+MNc_N^2\big)\cdot(1+N^2c_N^2).\end{aligned}$$

Hence, this gives us many possibilities:

M	a_N	b_N	c_N	$\|\mathbf{w}\|_2^2$	$\|\mathbf{w}\|_1$	Description
finite	$\frac{\beta^2}{M}$ const.	$\frac{\beta^2}{N}$	$\frac{\beta^2}{N^{\varepsilon+1/2}}$	$N^{1-2\varepsilon}$	$N^{1-\varepsilon}$	Macroscopic interaction by the first M spins, $\varepsilon>0$
N^α	$\frac{\beta^2}{M}=\frac{\beta^2}{N^\alpha}$	$\frac{\beta^2}{N}$	$\frac{\beta^2}{N^{\varepsilon+1+\alpha/2}}$	$N^{1-2\varepsilon}$	$N^{1/2}$	$M=o(N)$, $\alpha\in(0,1)$, $\varepsilon>0$
αN	$\frac{\beta^2}{M}=\frac{\beta^2}{\alpha N}$	$\frac{\beta^2}{N}$	$\frac{\beta^2}{N^{\varepsilon+3/4}}$	$N^{1-4\varepsilon}$	$N^{3/4}$	M is a fraction of spins, $\alpha\in(0,1)$, $\varepsilon\in(0,1/2)$

Actually, in the last example for $\varepsilon<\frac{1}{4}$, one has $\|\mathbf{w}\|_1=o(\sqrt{N})$. Remarkably, in the above examples, $\mathbf{w}$ exists for all c_N – given that $a_N<\frac{1}{L_0 M}$ and $b_N<\frac{1}{L_0 N}$. That is, even for $\varepsilon<0$, although then the convergences do not hold!

4.4. Self-Averaging of the Free Energy

For the sake of completeness, we here repeat the result from [18] that the free energy $\frac{1}{N}\log Z_N$ converges to its expectation if it exists. The existence of the expected free energy was proved only in some cases, c.f. Chapter 3.

First, we recall without proof the well-known concentration of measure for Gaussian fields. This is Theorem 1.3.3 in [16].

Proposition 4.7. *Let $\mathbf{g} = (g_1, \ldots, g_n)$ be i.i.d. standard normal random variables. Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz with constant L:*

$$|F(x) - F(y)| \leq L\|x - y\|_2.$$

Then, for all $u > 0$:

$$P\Big(|F(g) - \mathbb{E}F(g)| > u\Big) \leq 2\exp\left[\frac{-u^2}{4L^2}\right] \tag{4.18}$$

The following results were developed in [18]. Here, we give the generalization in the dilution setting.

Lemma 4.8. *Let $t \geq 0$ and $\mathbf{C}$ s.t. the matrix-norm $\|\mathbf{C}\|_1 = o(\sqrt{N})$. Then, there exists $L < \infty$ dependent only on Γ, s.t.:*

$$P\left(\left|\tfrac{1}{N}\log Z_N - \tfrac{1}{N}\mathbb{E}\log Z_N\right| > t\right) \leq 2\exp\left(-\frac{t^2 N}{L\|\mathbf{C}\|_1^2}\right).$$

Proof. Since Γ is positive semidefinite, there is a $|\Sigma^2| \times |\Sigma^2|$-matrix A such that:

$$\Gamma = A^T A.$$

Let z_{ij}, $i < j \leq N$, be an i.i.d. sequence of Gaussian random fields on Σ^2 with independent entries. Hence, we can represent

$$g_{ij}(s,t) = \sum_{s',t'} a(s,t,s',t')z_{ij}(s',t').$$

Consider:

$$\mathbf{z} = \big(z_{ij}(s_{ij}, t_{ij})\big)_{i<j, s_{ij}, t_{ij}}, \qquad \mathbf{a}(\boldsymbol{\sigma}) := \big(\sqrt{c_{ij}}\, a(\sigma_i, \sigma_j, s_{ij}, t_{ij})\big)_{i<j, s_{ij}, t_{ij}}$$

Then:

$$\sum_{i<j} \sqrt{c_{ij}}\, g_{ij}(\sigma_i, \sigma_j) = \sum_{i<j} \sum_{s'_{ij}, t'_{ij}} a(\sigma_i, \sigma_j, s'_{ij}, t'_{ij}) z_{ij}(s'_{ij}, t'_{ij}) = \mathbf{a}(\boldsymbol{\sigma}) \cdot \mathbf{z}$$

Now, due to our assumptions on $\mathbf{C}$, here we have that:

$$\begin{aligned} C_N := \|\mathbf{a}(\boldsymbol{\sigma})\|_2 = \sqrt{\sum_{i<j, s_{ij}, t_{ij}} c_{ij} a(\sigma_i, \sigma_j, s_{ij}, t_{ij})^2} &\leq |\Sigma| \cdot \sqrt{\sum_{i<j} c_{ij}} \cdot \|A\|_\infty \\ &\leq |\Sigma| \cdot \sqrt{N} \|\mathbf{C}\|_1 \cdot \|A\|_\infty \end{aligned}$$

Then, we have

$$\begin{aligned} \tfrac{1}{N} \log \mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{\mathbf{a}(\sigma)\cdot\mathbf{z}'} &\leq \tfrac{1}{N} \log \mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{\mathbf{a}(\sigma)\cdot\mathbf{z} + C_N \|\mathbf{z}' - \mathbf{z}\|_2} \\ &= \tfrac{1}{N} \log \mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{\mathbf{a}(\sigma)\cdot\mathbf{z}} + \tfrac{C_N}{N} \|\mathbf{z}' - \mathbf{z}\|_2 \end{aligned}$$

Therefore, $\mathbf{z} \mapsto \frac{1}{N} \log Z_N$ is Lipschitz continuous with constant $\frac{C_N}{N} \leq \frac{L\|\mathbf{C}\|_1}{\sqrt{N}}$. Then, concentration of measure in Proposition 4.7 gives:

$$P\left(\left|\tfrac{1}{N} \log Z_N - \tfrac{1}{N}\mathbb{E} \log Z_N\right| > t\right) \leq 2 \exp\left(-\frac{t^2 N}{2L^2 \|\mathbf{C}\|_1^2}\right) \qquad \square$$

Corollary 4.9. *The free energy* $\frac{1}{N} \log Z_N$ *is self-averaging, that is:*

$$\tfrac{1}{N} \log Z_N \xrightarrow[N\to\infty]{} \lim_{N\to\infty} \tfrac{1}{N} \mathbb{E} \log Z_N$$

in probability if $\|\mathbf{C}\|_1 = o(\sqrt{N})$.

5. Main Proofs

5.1. Main Lemma

Fix Γ, p, $\mathbf{C}$, and $\boldsymbol{\kappa}$. Both Theorems 4.3 and 4.4 are consequences of the same calculation. The main ingredient for our proofs is Lemma 5.1. In order to state it, we have to introduce the following notations. First, we will use the following suprema that are obviously linked to the left hand sides of Theorem 4.4, where $s, s' \in \Sigma$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N_{\geq 0}$:

$$
\begin{aligned}
U(\boldsymbol{\alpha}; s) &:= \sup_{H \in \mathcal{H}} \sqrt{\nu_H \Big\{ \big(\sum_i \alpha_i [\delta_i(s) - \pi_i(s)] \big)^2 \Big\}} \\
V(\boldsymbol{\alpha}; s, s') &:= \sup_{H \in \mathcal{H}} \sqrt{\nu_H \Big\{ \big(\sum_i \alpha_i [\delta_i^{1,2}(s, s') - \kappa_i(s, s')] \big)^2 \Big\}} \\
U(\boldsymbol{\alpha}) &:= \max_{\boldsymbol{\eta} \in \Sigma^N} \sup_{H \in \mathcal{H}} \sqrt{\nu_H \Big\{ \big(\sum_i \alpha_i [\delta_i(\eta_i) - \pi_i(\eta_i)] \big)^2 \Big\}} \\
V(\boldsymbol{\alpha}) &:= \max_{\boldsymbol{\eta}, \boldsymbol{\eta}' \in \Sigma^N} \sup_{H \in \mathcal{H}} \sqrt{\nu_H \Big\{ \big(\sum_i \alpha_i [\delta_i^{1,2}(\eta_i, \eta_i') - \kappa_i(\eta_i, \eta_i')] \big)^2 \Big\}}
\end{aligned}
$$

Trivially, $U(\boldsymbol{\alpha}; s) \leq U(\boldsymbol{\alpha})$ and $V(\boldsymbol{\alpha}; s, s') \leq V(\boldsymbol{\alpha})$ for all $s, s' \in \Sigma$. For any fixed $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N_{\geq 0}$, we will set for all $\boldsymbol{\sigma}, \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \boldsymbol{\eta}, \boldsymbol{\eta}' \in \Sigma^N$ and for all $i \leq N$:

$$
F_i(\boldsymbol{\eta}; \boldsymbol{\sigma}) := \big(\delta_i(\eta_i) - \pi_i(\eta_i)\big) \cdot \sum_{j=1}^N \alpha_j \big(\delta_j(\eta_j) - \pi_j(\eta_j)\big), \tag{5.1}
$$

$$
\begin{aligned}
G_i(\boldsymbol{\eta}, \boldsymbol{\eta}'; \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) := {}& \big(\delta_i^{1,2}(\eta_i, \eta_i') - \kappa_i(\eta_i, \eta_i')\big) \\
& \cdot \sum_{j=1}^N \alpha_j \big(\delta_j^{1,2}(\eta_j, \eta_j') - \kappa_j(\eta_j, \eta_j')\big)
\end{aligned}
$$

The use of F_i and G_i for our proofs becomes clearer after making the following observation:

$$\begin{aligned}\sum_{i=1}^N \alpha_i F_i(\boldsymbol{\eta};\boldsymbol{\sigma}) &= \sum_{i=1}^N \alpha_i\big(\delta_i(\eta_i)-\pi_i(\eta_i)\big)\cdot\sum_{j=1}^N \alpha_j\big(\delta_j(\eta_j)-\pi_j(\eta_j)\big)\\ &= \Big(\sum_{i=1}^N \alpha_i\big(\delta_i(\eta_i)-\pi_i(\eta_i)\big)\Big)^2.\end{aligned}$$

This should motivate our main lemma:

Lemma 5.1 (Main Lemma)**.** *Let $H = H_{\mathbf{A},\mathbf{B}} \in \mathcal{H}$. The following estimates hold for all $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_N) \in \mathbb{R}^N_{\geq 0}$ and for all $\boldsymbol{\eta},\boldsymbol{\eta}' \in \Sigma^N$:*

$$\begin{aligned}\left|\nu_H\Big[F_i(\boldsymbol{\eta};\boldsymbol{\sigma})\Big]\right| &\leq \alpha_i + 2\cdot K\cdot U(\boldsymbol{\alpha})\cdot\big(U(a_{\bullet i})+V(a_{\bullet i})\big),\\ \left|\nu_H\Big[G_i(\boldsymbol{\eta},\boldsymbol{\eta}';\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Big]\right| &\leq \alpha_i + 8\cdot K\cdot V(\boldsymbol{\alpha})\cdot\big(U(a_{\bullet i})+V(a_{\bullet i})\big),\end{aligned}$$

where $K := \sum_{s,t,s',t'\in\Sigma}|\Gamma(s,t,s',t')|$ and $a_{\bullet i}$ is the i-th row of $\mathbf{A}$ given by $H = H_{\mathbf{A},\mathbf{B}}$.

This lemma is proved using the smart path method by fading in the interaction with the i-th spin in the Hamiltonian H.

5.1.1. Proof of Lemma 5.1

We will define as usual a smart path $(H_x)_{x\in[0,1]} \subset \mathcal{H}$ to decouple the last spin from the others, and then use for functions of $\varphi(x) = \Phi(H_x)$:

$$\varphi(1) = \varphi(0) + \int_0^1 \varphi'(x)\,dx \leq \varphi(0) + \sup_x \varphi'(x)\,dx.$$

In order to handle the derivative, we will apply integration by parts and rewrite the result. Finally, Cauchy-Schwarz will bring the desired formula.

Fix $\boldsymbol{\eta},\boldsymbol{\eta}' \in \Sigma^N$ and $H = H_{\mathbf{A},\mathbf{B}} \in \mathcal{H}$. In order to get hold of U and V, we have to fade in all interaction with a spin, say the N-th. Therefore, we will write

$$F(\boldsymbol{\sigma}) := F_N(\boldsymbol{\eta};\boldsymbol{\sigma}), \qquad G(\boldsymbol{\sigma},\boldsymbol{\sigma}') := G_N(\boldsymbol{\eta},\boldsymbol{\eta}';\boldsymbol{\sigma},\boldsymbol{\sigma}').$$

We will further lighten our notation by writing:

$$\Phi^{(i)}(s) := \Phi_{\kappa_i}(s), \qquad \Gamma^{(i)}(s,s') := \Gamma_{\kappa_i}(s,s'), \qquad \gamma^{(i)}(s) := \Gamma^{(i)}(s,s).$$

We have to stay in $\mathcal{H}$, so we just give $a_{ij}(x), b_{ij}(x)$, $x \in [0,1]$:

$$\begin{aligned} a_{ij}(x) &:= \begin{cases} a_{ij} & i < j < N \\ x a_{ij} & i < j = N \end{cases}, \\ b_{ij}(x) &:= b_{ij} + \begin{cases} (1-x)a_{ij} & i = N \,\mathrm{or}\, j = N \\ 0 & i,j < N \end{cases} \end{aligned} \tag{5.2}$$

and let $H_x := H_{\mathbf{A}(x),\mathbf{B}(x)}$ and $\nu_x(\cdot) := \nu_{H_x}(\cdot)$. Now, consider:

$$\begin{aligned} \varphi_N(x) &:= \nu_x[F(\boldsymbol{\sigma})] = \nu_x[F_N(\boldsymbol{\eta};\boldsymbol{\sigma})], \\ \psi_N(x) &:= \nu_x[G(\boldsymbol{\sigma},\boldsymbol{\sigma}')] = \nu_x[G_N(\boldsymbol{\eta},\boldsymbol{\eta}';\boldsymbol{\sigma},\boldsymbol{\sigma}')]. \end{aligned}$$

Here, the index N indicates that we use F_N and G_N and fade out interaction with σ_N. It is clear that:

$$\begin{aligned} \varphi(0) &= \alpha_N \nu_0\Big[\big(\delta_N(\eta_N) - \pi_N(\eta_N)\big)^2\Big] \\ &\quad + \nu_0\Big[\big(\delta_N(\eta_N) - \pi_N(\eta_N)\big)\sum_{i<N}\alpha_i\big(\delta_i(\eta_i) - \pi_i(\eta_i)\big)\Big] \\ \psi(0) &= \alpha_N \nu_0\Big[\big(\delta_N^{1,2}(\eta_N,\eta_N') - \kappa_N(\eta_N,\eta_N')\big)^2\Big] \\ &\quad + \nu_0\Big[\big(\delta_N^{1,2}(\eta_N,\eta_N') - \kappa_N(\eta_N,\eta_N')\big)\sum_{i<N}\alpha_i\big(\delta_i^{1,2}(\eta_i,\eta_i') - \kappa_i(\eta_i,\eta_i')\big)\Big] \end{aligned}$$

In both cases the second summand vanishes. Indeed according to (5.2), the last spin is decoupled from the others. For $\psi(0)$, clearly κ_N fulfills the fixed point equation (4.4) in this case. For $\varphi(0)$ one sees that the fixed point equation for κ_N also implies:

$$\pi_N(\eta_N) = \sum_s \kappa_N(\eta_N, s) = \sum_s \nu_0[\delta_N^{1,2}(\eta_N, s)] = \nu_0[\delta_N(\eta_N)]$$

Differentiation and Integration by Parts First, we calculate the derivative of $\varphi(x)$. By elementary calculus and the definition of replicas and Gibbs probability, we have:

$$
\begin{aligned}
\varphi_N'(x) &= \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\, F(\boldsymbol{\sigma}) \frac{e^{H_x(\boldsymbol{\sigma})} \cdot H_x'(\boldsymbol{\sigma}) \cdot Z - e^{H_x(\boldsymbol{\sigma})} \cdot Z'}{Z^2} \\
&= \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2}\, F(\boldsymbol{\sigma}^1) \cdot \big(H_x'(\boldsymbol{\sigma}^1) - H_x'(\boldsymbol{\sigma}^2)\big) \cdot \frac{e^{H_x(\boldsymbol{\sigma}^1) + H_x(\boldsymbol{\sigma}^2)}}{Z^2} \\
&= \nu_x \left[F(\boldsymbol{\sigma}^1) \cdot \big(H_x'(\boldsymbol{\sigma}^1) - H_x'(\boldsymbol{\sigma}^2)\big) \right]
\end{aligned}
$$

Observe the introduction of a replica. The Hamiltonian differentiates w.r.t. x as follows:

$$
\begin{aligned}
H_x'(\boldsymbol{\sigma}) &= \frac{1}{2\sqrt{x}} \sum_{i<N} \sqrt{a_{iN}} g_{iN}(\sigma_i, \sigma_N) \\
&- \sum_{i<N} a_{iN} \left(\tfrac{1}{2\sqrt{b_{iN}(x)}} g_i^{(N)}(\sigma_i) + \tfrac{1}{2\sqrt{b_{iN}(x)}} g_N^{(i)}(\sigma_N) + \Phi^{(N)}(\sigma_i) + \Phi^{(i)}(\sigma_N) \right)
\end{aligned}
$$

Now, we use integration by parts

$$
\begin{aligned}
\nu_x[F(\boldsymbol{\sigma}) \cdot g_{iN}(\sigma_i, \sigma_N)] &= \sum \big(\mathbb{E} g g_{iN}(\sigma_i, \sigma_N)\big) \cdot \mathbb{E} \frac{\partial \langle\, F(\boldsymbol{\sigma}) \,\rangle}{\partial g} \\
&= \sum \big(\mathbb{E} g g_{iN}(\sigma_i, \sigma_N)\big) \cdot \mathbb{E} \frac{\partial}{\partial g}\, \mathrm{Tr}_{\boldsymbol{\sigma}}\, F(\boldsymbol{\sigma})\, \frac{e^{H(\boldsymbol{\sigma})}}{Z_N},
\end{aligned}
$$

where the sums are over all Gaussians g that appear in the expression. Fortunately, $F(\boldsymbol{\sigma})$ does not depend on any Gaussians. Therefore, we only have the $g_{iN}(\sigma_i, \sigma_N)$ in the numerator and the $g_{iN}(\sigma_i', \sigma_N')$ in the denominator of the Gibbs probability. Thus we have:

$$
\begin{aligned}
&\nu_x[F(\boldsymbol{\sigma}) \cdot g_{iN}(\sigma_i, \sigma_N)] \\
&\quad= \sqrt{x a_{iN}}\, \nu_x \left[F(\boldsymbol{\sigma}^1) \cdot \big(\mathbb{E} g_{iN}(\sigma_i, \sigma_N)^2 - \mathbb{E} g_{iN}(\sigma_i, \sigma_N) g_{iN}(\sigma_i', \sigma_N') \big) \right] \\
&\quad= \sqrt{x a_{iN}}\, \nu_x \left[F(\boldsymbol{\sigma}^1) \cdot \big(\gamma(\sigma_i, \sigma_N) - \Gamma^\star(\sigma_i, \sigma_N; \sigma_i', \sigma_N') \big) \right]
\end{aligned}
$$

The minus of the second summand is due to the fact that this second replica stems from the sum in the denominator, i.e. has power -1. In cases where the expression consists of n replicas, the replica ascending from the denominator has power $-n$, as we will see in what follows. Now, we substitute this calculation in the following expression:

$$\begin{aligned}
&\tfrac{1}{2\sqrt{x}}\,\nu_x\big[F(\boldsymbol{\sigma}^1)\cdot\sum_i\sqrt{a_{ij}}g_{iN}(\sigma_i^1,\sigma_N^1)\big] && (5.3)\\
&\quad=\tfrac{1}{2}\sum_i a_{iN}\nu_x\big[F(\boldsymbol{\sigma}^1)\cdot\big(\gamma(\sigma_i^1,\sigma_N^1)-\Gamma(\sigma_i^1,\sigma_N^1,\sigma_i^2,\sigma_N^2)\big)\big]\\
&\quad=\tfrac{1}{2}\sum_i a_{iN}\nu_x\big[F(\boldsymbol{\sigma}^1)\cdot\big(\gamma(\sigma_i^1,\sigma_N^1)+\Gamma(\sigma_i^1,\sigma_N^1,\sigma_i^2,\sigma_N^2)\big) && (5.4)\\
&\qquad\qquad-2\Gamma(\sigma_i^1,\sigma_N^1,\sigma_i^3,\sigma_N^3)\big)\big]
\end{aligned}$$

The reason for introducing the last equality will become clear in a moment. It holds because of symmetry of replicas which are not used in the rest of the expression. Proceeding by the same integration by parts as above, we obtain for the following expression:

$$\begin{aligned}
&\tfrac{1}{2\sqrt{x}}\,\nu_x\big[F(\boldsymbol{\sigma}^1)\cdot\sum_i\sqrt{a_{iN}}g_{iN}(\sigma_i^2,\sigma_N^2)\big]\\
&\quad=\tfrac{1}{2}\sum_i a_{iN}\nu_x\big[F(\boldsymbol{\sigma}^1)\cdot\big(\gamma(\sigma_i^2,\sigma_N^2)+\Gamma(\sigma_i^2,\sigma_N^2,\sigma_i^1,\sigma_N^1)\big)\\
&\qquad\qquad-2\Gamma(\sigma_i^2,\sigma_N^2,\sigma_i^3,\sigma_N^3)\big)\big]
\end{aligned}$$

Observe that in the expression on the left hand side we have already two replicas involved. Therefore, there are three Gaussians in this expression that are not independent of g_{iN}: Two in the numerator of the Gibbs probability, giving the first two summands, and one in the square of the denominator, giving the third summand with the factor -2. We inserted the additional terms in (5.3) to make it similar to this one here.

Similar calculations are performed for the Gaussians $g_i^{(N)}$ and $g_N^{(i)}$. Hence, we now get for the derivative of $\varphi(x)$:

$$\begin{aligned}
\varphi_N'(x) &= \nu_x\left[F(\boldsymbol{\sigma}^1)\left\{H_x'(\boldsymbol{\sigma}^1) - H_x'(\boldsymbol{\sigma}^2)\right\}\right] \\
&= \tfrac{1}{2}\nu_x\Bigg[F(\boldsymbol{\sigma}^1)\sum_{i<N} a_{iN}\Bigg\{\gamma(\sigma_i^1,\sigma_N^1) - \gamma(\sigma_i^2,\sigma_N^2) \\
&\qquad - 2\Gamma(\sigma_i^1,\sigma_N^1,\sigma_i^3,\sigma_N^3) + 2\Gamma(\sigma_i^2,\sigma_N^2,\sigma_i^3,\sigma_N^3) \\
&\qquad + \Gamma(\sigma_i^1,\sigma_N^1,\sigma_i^2,\sigma_N^2) - \Gamma(\sigma_i^1,\sigma_N^1,\sigma_i^2,\sigma_N^2) \\
&\qquad - \Bigg[\gamma^{(N)}(\sigma_i^1) - \gamma^{(N)}(\sigma_i^2) - 2\Gamma^{(N)}(\sigma_i^1,\sigma_i^3) + 2\Gamma^{(N)}(\sigma_i^2,\sigma_i^3) \\
&\qquad\quad + \gamma^{(i)}(\sigma_N^1) - \gamma^{(i)}(\sigma_N^2) - 2\Gamma^{(i)}(\sigma_N^1,\sigma_N^3) + 2\Gamma^{(i)}(\sigma_N^2,\sigma_N^3) \\
&\qquad\quad + 2\Phi^{(i)}(\sigma_N^1) + 2\Phi^{(N)}(\sigma_i^1) - 2\Phi^{(i)}(\sigma_N^2) - 2\Phi^{(N)}(\sigma_i^2)\Bigg]\Bigg\}\Bigg]
\end{aligned}$$

By collecting terms, this becomes

$$\varphi_N'(x) = \tfrac{1}{2}\nu_x\Bigg[F(\boldsymbol{\sigma}^1)\big(v(1) - v(2) - 2v(1,2) + 2v(2,3)\big)\Bigg],$$

where we use the notation for $l \neq l'$:

$$\begin{aligned}
v(l) &:= \sum_{i<N} a_{iN}\Big[\gamma(\sigma_i^l,\sigma_N^l) - \gamma(\sigma_i^l,\pi_N) - \gamma(\sigma_N^l,\pi_i)\Big], \\
v(l,l') &:= \sum_{i<N} a_{iN}\Big[\Gamma(\sigma_i^l,\sigma_N^l,\sigma_i^{l'},\sigma_N^{l'}) - \Gamma^{(N)}(\sigma_i^l,\sigma_i^{l'}) - \Gamma^{(i)}(\sigma_N^l,\sigma_N^{l'})\Big]
\end{aligned}$$

Note that the $\gamma^{(\bullet)}(s) = \Gamma^\star(s,s;\kappa_\bullet)$ in the differentiation were converted by the $\Phi^{(\bullet)}$ terms to $\gamma(s,\pi_\bullet) = \sum_t \gamma(s,\pi_\bullet(t))$.

Now, for ψ_N we again obtain by simple calculus:

$$\psi_N'(x) = \nu_x\left[G(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\left\{H_x'(\boldsymbol{\sigma}^1) + H_x'(\boldsymbol{\sigma}^2) - 2H_x'(\boldsymbol{\sigma}^3)\right\}\right]$$

We handle this once more using integration by parts. Of course, since we already have here three replicas, this will introduce a fourth one with the appropriate factor. We spare the reader the details of the derivation, since it is straightforward:

$$\begin{aligned}
\psi_N'(x) = \tfrac{1}{2}\nu_x\Bigg[G(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\sum_{i<N} a_{iN}\Big\{&\gamma(\sigma_i^1,\sigma_N^1)+\gamma(\sigma_i^2,\sigma_N^2)-2\gamma(\sigma_i^3,\sigma_N^3)\\
&+\gamma^{(N)}(\sigma_i^1)+\gamma^{(N)}(\sigma_i^2)-2\gamma^{(N)}(\sigma_i^3)+\gamma^{(i)}(\sigma_N^1)+\gamma^{(i)}(\sigma_N^2)-2\gamma^{(i)}(\sigma_N^3)\\
&+2\Phi^{(i)}(\sigma_N^1)+2\Phi^{(N)}(\sigma_i^1)+2\Phi^{(i)}(\sigma_N^2)+2\Phi^{(N)}(\sigma_i^2)\\
&-4\Phi^{(i)}(\sigma_N^3)-4\Phi^{(N)}(\sigma_i^3)\Big\}\Bigg]\\
+\tfrac{1}{2}\nu_x\Bigg[G(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\cdot\big\{&2v(1,2)-v(1,3)-v(2,3)\\
&-3v(1,4)-3v(2,4)+6v(3,4)\big\}\Bigg]
\end{aligned}$$

Again, by collecting terms and using symmetry between replicas, we obtain:

$$\begin{aligned}
\psi_N'(x) = \tfrac{1}{2}\nu_x\Bigg[G(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Bigg(&v(1)+v(2)-2v(3)\\
&+2v(1,2)-4v(1,3)-4v(2,3)+6v(3,4)\Bigg)\Bigg]
\end{aligned}$$

Reordering of terms Now that we know the derivatives of φ_N and ψ_N, we will rewrite them slightly. Observe that we can state $v(l,l')$ for $l \neq l'$ as:

$$\begin{aligned}
v(l,l') = \sum_{i<N} a_{iN}\sum_{s,t,s',t'}\Gamma(s,t,s,'t')\cdot\Big[&\big(\delta_i^{l,l'}(s,s')-\kappa_i(s,s')\big)\\
&\cdot\big(\delta_N^{l,l'}(t,t')-\kappa_N(t,t')\big)-\kappa_i(s,s')\kappa_N(t,t')\Big]
\end{aligned}$$

The last summand does not depend on $\boldsymbol{\sigma}^l$ and cancels if we e.g. have something like $v(1,2) - v(2,3)$. Therefore, if we write for $l \neq l'$:

$$\bar{v}(l) := \sum_{i<N} a_{iN} \sum_{s,t} \gamma(s,t) \Big[\big(\delta_i^l(s) - \pi_i(s)\big) \cdot \big(\delta_N^l(t) - \pi_N(t)\big) \Big]$$

$$\begin{aligned} \bar{v}(l,l') := \sum_{i<N} a_{iN} \sum_{s,t,s',t'} \Gamma(s,t,s,'t') \Big[&\big(\delta_i^{l,l'}(s,s') - \kappa_i(s,s')\big) \\ &\big(\delta_N^{l,l'}(t,t') - \kappa_N(t,t')\big) \Big], \end{aligned}$$

because all $v(\cdot)$ are balanced out, we obtain:

$$\begin{aligned} \varphi_N'(x) &= \tfrac{1}{2}\nu_x \Big[F(\boldsymbol{\sigma}^1) \Big(\bar{v}(1) - \bar{v}(2) - 2\bar{v}(1,2) + 2\bar{v}(2,3) \Big) \Big], \\ \psi_N'(x) &= \tfrac{1}{2}\nu_x \Big[G(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \Big(\bar{v}(1) + \bar{v}(2) - 2\bar{v}(3) \\ &\qquad + 2\bar{v}(1,2) - 4\bar{v}(1,3) - 4\bar{v}(2,3) + 6\bar{v}(3,4) \Big) \Big]. \end{aligned}$$

Use the following notation for $l \neq l'$:

$$\begin{aligned} F(l,l') &:= \tfrac{1}{2}\nu_x \big[F(\boldsymbol{\sigma}^1) \cdot \bar{v}(l,l') \big], & F(l) &:= \tfrac{1}{2}\nu_x \big[F(\boldsymbol{\sigma}^1) \cdot \bar{v}(l) \big], \\ G(l,l') &:= \tfrac{1}{2}\nu_x \big[G(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \cdot \bar{v}(l,l') \big], & G(l) &:= \tfrac{1}{2}\nu_x \big[G(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \cdot \bar{v}(l) \big]. \end{aligned}$$

Expanding this definition, we have for $l \neq l'$:

$$\begin{aligned}
F(l,l') = \tfrac{1}{2} \sum_{s,t,s',t'} \Gamma(s,t,s',t')\, \nu_x \Bigg[\qquad\qquad (5.5)\\
\cdot \big(\delta_N^1(\eta_N) - \pi_N(\eta_N)\big) \sum_{i \le N} \alpha_i \big(\delta_i^1(\eta_i) - \pi_i(\eta_i)\big) \\
\cdot \big(\delta_N^{l,l'}(t,t') - \kappa_N(t,t')\big) \sum_{i<N} a_{iN} \big(\delta_i^{l,l'}(s,s') - \kappa_i(s,s')\big) \Bigg], \\
G(l,l') = \tfrac{1}{2} \sum_{s,t,s',t'} \Gamma(s,t,s',t')\, \nu_x \Bigg[\\
\big(\delta_N^{1,2}(\eta_N,\eta_N') - \kappa_N(\eta_N,\eta_N')\big) \sum_{i \le N} \alpha_i \big(\delta_i^{1,2}(\eta_i,\eta_i') - \kappa_i(\eta_i,\eta_i')\big) \\
\cdot \big(\delta_N^{l,l'}(t,t') - \kappa_N(t,t')\big) \sum_{i<N} a_{iN} \big(\delta_i^{l,l'}(s,s') - \kappa_i(s,s')\big) \Bigg].
\end{aligned}$$

In both expressions, we have the product of two F_i and G_i type expressions, respectively, in the big brackets. We will use the Cauchy-Schwarz inequality for this product.

Cauchy-Schwarz Obviously, $|\delta_N^l(\eta_i) - \pi_N(\eta_i)| \le 1$ and the same also holds for the first factors in $G(l,l')$, $F(l)$ and $G(l)$. Thus, using $\sum |\Gamma(s,t,s',t')| \le K$ and Cauchy-Schwarz inequality, we have for all $l \neq l'$:

$$\begin{aligned}
|F(l)| &\le \frac{1}{2} \sum_{s,t} \gamma(s,t) \cdot U(\boldsymbol{\alpha})\, U(\mathbf{a}_{\bullet N}; s) && \le \tfrac{K}{2} U(\boldsymbol{\alpha}) \cdot U(\mathbf{a}_{\bullet N}), \\
|F(l,l')| &\le \frac{1}{2} \sum_{s,t,s',t'} |\Gamma(s,t,s',t')| \cdot U(\boldsymbol{\alpha})\, V(\mathbf{a}_{\bullet N}; s, s') && \le \tfrac{K}{2} U(\boldsymbol{\alpha}) \cdot V(\mathbf{a}_{\bullet N}), \\
|G(l)| &\le \frac{1}{2} \sum_{s,t} \gamma(s,t) \cdot V(\boldsymbol{\alpha})\, U(\mathbf{a}_{\bullet N}; s) && \le \tfrac{K}{2} V(\boldsymbol{\alpha}) \cdot U(\mathbf{a}_{\bullet N}), \\
|G(l,l')| &\le \frac{1}{2} \sum_{s,t,s',t'} |\Gamma(s,t,s',t')| \cdot V(\boldsymbol{\alpha})\, V(\mathbf{a}_{\bullet N}; s, s') && \le \tfrac{K}{2} V(\boldsymbol{\alpha}) \cdot V(\mathbf{a}_{\bullet N}),
\end{aligned}$$

Substituting this into the formula for the derivatives, they become:

$$
\begin{aligned}
|\varphi_N'(x)| &\le \tfrac{K}{2}\, U(\boldsymbol{\alpha}) \Big[2U(\mathbf{a}_{\bullet N}) + 4V(\mathbf{a}_{\bullet N}) \Big] \\
&\le 2 \cdot K \cdot U(\boldsymbol{\alpha}) \cdot \Big(U(\mathbf{a}_{\bullet N}) + V(\mathbf{a}_{\bullet N}) \Big), \\
|\psi_N'(x)| &\le \tfrac{K}{2}\, U(\boldsymbol{\alpha}) \Big[4U(\mathbf{a}_{\bullet N}) + 16V(\mathbf{a}_{\bullet N}) \Big] \\
&\le 8 \cdot K \cdot V(\boldsymbol{\alpha}) \cdot \Big(U(\mathbf{a}_{\bullet N}) + V(\mathbf{a}_{\bullet N}) \Big).
\end{aligned}
$$

Summarizing things up, we obtain:

$$
\begin{aligned}
|\varphi_N(1)| &\le \alpha_N + 2 \cdot K \cdot U(\alpha) \cdot \Big(U(\mathbf{a}_{\bullet N}) + V(\mathbf{a}_{\bullet N}) \Big), \\
|\psi_N(1)| &\le \alpha_N + 8 \cdot K \cdot V(\alpha) \cdot \Big(U(\mathbf{a}_{\bullet N}) + V(\mathbf{a}_{\bullet N}) \Big).
\end{aligned}
$$

This proves Lemma 5.1. □

5.2. Proofs of Theorems 4.3 and 4.4

We have to investigate the bounds given in the Main Lemma 5.1. Nicely enough, the Main Lemma can be used on itself again and again recursively. Indeed,

$$
\begin{aligned}
U(\boldsymbol{\alpha})^2 &= \sup_{H,\boldsymbol{\eta}} \sum_i \alpha_i \nu_H \big[F_i(\boldsymbol{\eta}; \boldsymbol{\sigma}) \big] \\
V(\boldsymbol{\alpha})^2 &= \sup_{H,\boldsymbol{\eta},\boldsymbol{\eta}'} \sum_i \alpha_i \nu_H \big[G_i(\boldsymbol{\eta}, \boldsymbol{\eta}'; \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) \big].
\end{aligned}
$$

Setting

$$
v_i := \tfrac{1}{2} \Big(\sup U(\boldsymbol{\beta}) + \sup V(\boldsymbol{\beta}) \Big), \qquad \forall i \le N,
$$

where the suprema are over all sequences $\boldsymbol{\beta}$ s.t. $0 \le \beta_j \le c_{ij}$, Lemma 5.1 gives:

$$U(\boldsymbol{\alpha})^2 \le \sum_i \alpha_i^2 + 4K \cdot U(\boldsymbol{\alpha}) \sum_i \alpha_i v_i$$
$$V(\boldsymbol{\alpha})^2 \le \sum_i \alpha_i^2 + 16 \cdot K \cdot V(\boldsymbol{\alpha}) \sum_i \alpha_i v_i$$

since the right hand sides do not depend on $\boldsymbol{\eta}$ or $\boldsymbol{\eta}'$. Using the fact that $x^2 \le Ax + B$ implies $x \le A + \sqrt{B}$ we get the uniform inequality:

$$\max\{U(\boldsymbol{\alpha}), V(\boldsymbol{\alpha})\} \le \sqrt{\sum_i \alpha_i^2} + 16 \cdot K \cdot \sum_i \alpha_i v_i. \tag{5.6}$$

Now we apply this recursively on the v_i:

Corollary 5.2. *We have for all $i \le N$:*

$$v_i \le w_i,$$

where $\mathbf{w} = (w_1, \dots, w_N)$ has been defined in (4.8)*.*

Proof. We apply (5.6) for all rows $\mathbf{c}_{\bullet,i}$, $i \le N$, of $\mathbf{C}$. Let $\mathbf{v}$ be the vector with components v_i and set:

$$L_0 := 16 \cdot K = 16 \sum_{s,t,s',t'} |\Gamma(s,t,s',t')| \tag{5.7}$$

Then, since U and V are non-negative, (5.6) gives $\mathbf{v} \le \mathbf{b} + L_0 \cdot \mathbf{C} \cdot \mathbf{v}$, where the inequality is componentwise. Recall that $b_i = \|c_{\bullet,i}\|_2$. Then, we get recursively:

$$\begin{aligned}\mathbf{v} &\le \mathbf{b} + L_0\mathbf{C}\mathbf{v} \le \mathbf{b} + L_0\mathbf{C}\mathbf{b} + (L_0\mathbf{C})^2\mathbf{v} \\ &\le \mathbf{b} + L_0\mathbf{C}\mathbf{b} + (L_0\mathbf{C})^2\mathbf{b} + (L_0\mathbf{C})^3\mathbf{v} \\ &\le \left(\sum_{i=0}^{\infty} (L_0\mathbf{C})^i\right) \cdot \mathbf{b} = \mathbf{M} \cdot \mathbf{b} = \mathbf{w}.\end{aligned}$$ □

Proof of Theorem 4.4. Just apply the previous corollary on (5.6). □

Another consequence of the above calculation is the following:

Lemma 5.3. *For each $i \leq N$ we have uniformly over all $H \in \mathcal{H}$ and all $s, t, s', t' \in \Sigma$:*

$$\Big|\nu_H\Big[\big(\delta_i(s) - \pi_i(s)\big)\sum_j a_{ij}\big(\delta_j(t) - \pi_j(t)\big)\Big]\Big| \leq Lw_i^2$$

$$\Big|\nu_H\Big[\big(\delta_i^{1,2}(s,s') - \kappa_i(s,s')\big)\sum_j a_{ij}\big(\delta_j^{1,2}(t,t') - \kappa_j(t,t')\big)\Big]\Big| \leq Lw_i^2$$

Proof. Fix $i \leq N$ and $H \in \mathcal{H}$. We set $\eta_i := s$, $\eta_i' := s'$ and for $j \neq i$ let $\eta_j := t$, $\eta_j' := t'$. Then, we use the above calculations for $\alpha_j := a_{ij}$ to obtain:

$$\big(\delta_i(s) - \pi_i(s)\big)\sum_j a_{ij}\big(\delta_j(t) - \pi_j(t)\big) = F_i(\boldsymbol{\eta};\boldsymbol{\sigma}),$$

$$\big(\delta_i^{1,2}(s,s') - \kappa_i(s,s')\big)\sum_j a_{ij}\big(\delta_j^{1,2}(t,t') - \kappa_j(t,t')\big) = G_i(\boldsymbol{\eta},\boldsymbol{\eta}';\boldsymbol{\sigma},\boldsymbol{\sigma}')$$

Observe that the terms with $j = i$ vanish since $\alpha_i = a_{ii} = 0$. Therefore, we can apply the Main Lemma 5.1 to bound the right hand sides above:

$$\left|\nu_H\Big[F_i(\boldsymbol{\eta};\boldsymbol{\sigma})\Big]\right| \leq 2\cdot K\cdot U(a_{\bullet i})\cdot\big(U(a_{\bullet i}) + V(a_{\bullet i})\big) \leq 4Kw_i^2,$$

$$\left|\nu_H\Big[G_i(\boldsymbol{\eta},\boldsymbol{\eta}';\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\Big]\right| \leq 8\cdot K\cdot V(a_{\bullet i})\cdot\big(U(a_{\bullet i}) + V(a_{\bullet i})\big) \leq 16Kw_i^2. \quad \square$$

Using this lemma, we get our main result.

Proof of Theorem 4.3. We look at:

$$H_x(\boldsymbol{\sigma}) := \sum_{i<j}\sqrt{xa_{ij}}\,g_{ij}(\sigma_i,\sigma_j) + \sum_{i,j}\sqrt{b_{ij} + (1-x)a_{ij}}\,g_i^{(j)}(\sigma_i)$$
$$+ \sum_{i,j}\big(b_{ij} + (1-x)a_{ij}\big)\Phi^{(j)}(\sigma_i).$$

That is, we decouple every interaction in the system. Obviously, $H_x \in \mathcal{H}$ for all $x \in [0,1]$. Let $\varphi(x) := \mathbb{E}\log(\mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{H_x(\boldsymbol{\sigma})})$. As usual, we differentiate this w.r.t. x. First, we have:

$$H_x'(\boldsymbol{\sigma}) = \sum_{i<j}\tfrac{\sqrt{a_{ij}}}{2\sqrt{x}}g_{ij}(\sigma_i,\sigma_j) - \sum_{i,j}\tfrac{a_{ij}}{2\sqrt{b_{ij}+(1-x)a_{ij}}}g_i^{(j)}(\sigma_i) - \sum_{i,j}a_{ij}\Phi^{(j)}(\sigma_i).$$

Now, we can calculate:

$$
\begin{aligned}
\varphi'(x) &= \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\,\frac{e^{H_x(\boldsymbol{\sigma})}\cdot H_x'(\boldsymbol{\sigma})}{Z} = \nu_x\Big[H_x'(\boldsymbol{\sigma})\Big] \\
&= \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j') - \gamma^{(j)}(\sigma_i) + \Gamma^{(j)}(\sigma_i,\sigma_i') \\
&\qquad - \gamma^{(i)}(\sigma_j) + \Gamma^{(i)}(\sigma_j,\sigma_j') - 2\Phi^{(i)}(\sigma_j) - 2\Phi^{(j)}(\sigma_i)\Big]
\end{aligned}
$$

Expanding the definitions of $\Gamma^{(j)}(s,s') = \Gamma_{\kappa_j}(s,s')$ and using the $\Phi^{(\cdot)}$ to translate, we get:

$$
\begin{aligned}
\varphi'(x) &= \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j') + \Gamma^\star(\sigma_i,\sigma_i';\kappa_j) \\
&\qquad + \Gamma^\star(\sigma_j,\sigma_j';\kappa_i) - \gamma(\sigma_j,\pi_i) - \gamma(\sigma_i,\pi_j)\Big]. \\
&= \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \gamma(\sigma_j,\pi_i) - \gamma(\sigma_i,\pi_j) + \gamma(\pi_i,\pi_j) \\
&\qquad - \Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j') + \Gamma^\star(\sigma_i,\sigma_i';\kappa_j) + \Gamma^\star(\sigma_j,\sigma_j';\kappa_i) - \Gamma^\star(\kappa_i;\kappa_j)\Big] \\
&\qquad - \frac{1}{4}\sum_{i,j} a_{ij}\big(\gamma(\pi_i,\pi_j) - \Gamma^\star(\kappa_i;\kappa_j)\big).
\end{aligned}
$$

Here, we see that the first term in the final formula has appeared on stage.

We rewrite this as:

$$\begin{aligned}
\varphi'(x) &= \frac{1}{4}\sum_{s,t}\gamma(s,t)\sum_{ij}a_{ij}\nu_x\Big[\big(\delta_i(s)-\pi_i(s)\big)\big(\delta_j(t)-\pi_j(t)\big)\Big] \\
&\quad -\frac{1}{4}\sum_{s,t,s',t'}\Gamma(s,t,s',t')\sum_{ij}a_{ij}\nu_x\Big[\big(\delta_i^{1,2}(s,s')-\kappa_i(s,s')\big) \\
&\qquad\qquad \big(\delta_j^{1,2}(t,t')-\kappa_j(t,t')\big)\Big]-\frac{1}{4}\sum_{i,j}a_{ij}\big(\gamma(\pi_i,\pi_j)-\Gamma^\star(\kappa_i;\kappa_j)\big) \\
&= \frac{1}{4}\sum_{s,t}\gamma(s,t)\sum_{i}\nu_x\Big[\big(\delta_i(s)-\pi_i(s)\big)\sum_j a_{ij}\big(\delta_j(t)-\pi_i(t)\big)\Big] \\
&\quad -\frac{1}{4}\sum_{s,t,s',t'}\Gamma(s,t,s',t')\sum_{i}\nu_x\Big[\big(\delta_i^{1,2}(s,s')-\kappa_i(s,s')\big) \\
&\qquad\qquad \sum_j a_{ij}\big(\delta_j^{1,2}(t,t')-\kappa_j(t,t')\big)\Big]-\frac{1}{4}\sum_{i,j}a_{ij}\big(\gamma(\pi_i,\pi_j)-\Gamma^\star(\kappa_i;\kappa_j)\big).
\end{aligned}$$

Now:

$$\varphi(0)=\sum_{i\le N}\mathbb{E}\log\mathrm{Tr}_s\exp\big(Y_i(s)+\Phi^{(\mathbf{C},i)}(s)\big)$$

Since $H_x\in\mathcal{H}$ for all $x\in[0,1]$, we can apply the uniform bound of Lemma 5.3, which finishes the proof. $\square$

5.3. Proofs on Asymptotic Independence of Finite Number of Spins

Proof of Proposition 4.5. Fix $H\in\mathcal{H}$ and remove it from the notation. Set $\dot\delta_i:=\delta_i(\eta_i)-\mu_i(\eta_i)$. We calculate:

$$\begin{aligned}
\sum_{i<j}\alpha_i\alpha_j\mathbb{E}\Big[\Big\langle\ \dot\delta_i\cdot\dot\delta_j\ \Big\rangle^2\Big] &= \sum_{i<j}\alpha_i\alpha_j\nu\Big[\dot\delta_i^1\cdot\dot\delta_i^2\cdot\dot\delta_j^1\cdot\dot\delta_j^2\Big] \\
&\le \frac{1}{2}\nu\Big[\Big(\sum_i\alpha_i\dot\delta_i^1\cdot\dot\delta_i^2\Big)^2\Big],
\end{aligned}$$

where we added the non-negative diagonal terms in the inequality. For the summands in this sum, we now have:

$$\begin{aligned}\dot{\delta}_i^1 \cdot \dot{\delta}_i^2 &= \delta_i^{12}(\eta_i,\eta_i) - \delta_i^1(\eta_i)\mu_i(\eta_i) - \mu_i(\eta_i)\delta_i^2(\eta_i) + \mu_i(\eta_i)^2 \\ &= \big(\delta_i^{12}(\eta_i,\eta_i) - \kappa_i(\eta_i,\eta_i)\big) - \big(\delta_i^1(\eta_i)\mu_i(\eta_i) - \kappa_i(\eta_i,\eta_i)\big) \\ &\qquad - \big(\mu_i(\eta_i)\delta_i^2(\eta_i) - \kappa_i(\eta_i,\eta_i)\big) + \big(\mu_i(\eta_i)^2 - \kappa_i(\eta_i,\eta_i)\big).\end{aligned}$$

Here, we simply used the κ as approximations. Hence, the proof follows by using $(a+b+c+d)^2 \le 4\,(a^2+b^2+c^2+d^2)$, Jensen's inequality, and Theorem 4.4. $\square$

In the setting of Theorem 4.6, this proposition gives decorrelation of any pair of spins, say for σ_1 and σ_2. We will use this observation as the first step of an induction in the following proof.

Proof of Theorem 4.6. By definition of total variation and Cauchy-Schwarz:

$$|P_{1\dots n} - \mu_n|^2 = \left(\sum_{\boldsymbol{\eta}\in\Sigma^n} |P_{1\dots n}(\{\boldsymbol{\eta}\}) - \boldsymbol{\mu}_n(\{\boldsymbol{\eta}\})|\right)^2$$

$$= \left(\sum_{\boldsymbol{\eta}\in\Sigma^n} |\langle\, \boldsymbol{\delta}(\boldsymbol{\eta})\,\rangle - \prod_{i\le n}\mu_i(\eta_i)|\right)^2 \le 2^n \sum_{\boldsymbol{\eta}\in\Sigma^n}\left(\langle\, \boldsymbol{\delta}(\boldsymbol{\eta})\,\rangle - \prod_{i\le n}\mu_i(\eta_i)\right)^2,$$

where $\boldsymbol{\delta}(\boldsymbol{\eta}) := \prod_{i=1}^n \delta_i(\eta_i)$. Therefore, it suffices to show for any $\boldsymbol{\eta} \in \Sigma^n$, which we drop in the notation, that:

$$\mathbb{E}\left(\langle\, \boldsymbol{\delta}\,\rangle - \prod_{i\le n}\mu_i(\eta_i)\right)^2 \le \frac{L}{N} + \frac{L}{N^2}\left(\sum_i w_i\right)^2$$

This will be proved by induction over n for an $\boldsymbol{\eta} \in \Sigma^N$ fixed from now on. The case $n = 2$ was handled by the previous proposition. For the induction from $n-1$ to n, we show that:

$$\begin{aligned}\mathbb{E}\Big(\langle\, \boldsymbol{\delta}\,\rangle - \langle\, \delta_1\cdots\delta_{n-1}\,\rangle\langle\, \delta_n\,\rangle\Big)^2 &= \mathbb{E}\Big(\Big\langle\, \delta_1\cdots\delta_{n-1}\dot{\delta}_n\,\Big\rangle\Big)^2 \\ &\le \frac{L}{N} + \frac{L}{N^2}\left(\sum_i w_i\right)^2,\end{aligned}$$

where we use the notation $\dot{\delta}_i := \delta_i - \langle\, \delta_i \,\rangle$, $i \leq n$. Now:

$$\begin{aligned}
\frac{1}{N}\sum_i \mathbb{E}\Big(\Big\langle\, \delta_1 \cdots \delta_{n-1}\dot{\delta}_i \,\Big\rangle^2\Big) &= \frac{1}{N}\sum_i \nu\Big[\delta_1^{12}\cdots\delta_{n-1}^{12}\dot{\delta}_i^{12}\Big] \\
&= \nu\Big[\delta_1^{12}\cdots\delta_{n-1}^{12}\frac{1}{N}\sum_i \dot{\delta}_i^{12}\Big] \\
&\leq \sqrt{\nu\Big[\delta_1^{12}\cdots\delta_{n-1}^{12}\Big]^2}\cdot\sqrt{\nu\Big[\frac{1}{N}\sum_i \dot{\delta}_i^{12}\Big]^2}
\end{aligned}$$

Here, the first factor is bound by 1 and the second factor is due to Theorem 4.4 of the right order, which proves the induction. □

5.4. Uniqueness of $\boldsymbol{\kappa}$ at High Enough Temperature

To finish this chapter, we give the generalization of a proof in [18] for the dilution setting:

Proof of Lemma 4.2 (b). We have to calculate the Lipschitz constant of the following function to see that it is a contraction:

$$\boldsymbol{\kappa} = \big(\kappa_i(\cdot,\cdot)\big)_{i\leq N} \mapsto \big(\mathbb{E}\,\Pi_{\boldsymbol{\kappa}}^{(\mathbf{C},i)}(\cdot)\cdot\Pi_{\boldsymbol{\kappa}}^{(\mathbf{C},i)}(\cdot)\big)_{i\leq N}$$

Let $\boldsymbol{\kappa} = (\kappa_i)_{i\leq N}$ and $\boldsymbol{\kappa}' = (\kappa_i')_{i\leq N}$ be two sequences of matrices. We will need the following:

$$D_i := \sum_{j\leq N} c_{ij}\|\kappa_j - \kappa_j'\|_\infty = \sum_{j\leq N} c_{ij}\max_{s,s'}|\kappa_j(s,s') - \kappa_j'(s,s')|$$

We fix $i \leq N$ to get a bound on the i-th entry. We do this as usual by interpolating by a parameter $x \in [0,1]$ along the obvious smart path between the independent systems corresponding to the local field Hamiltonian of the i-th spin under $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$, respectively:

$$\begin{aligned}
H_x(s) &:= \sqrt{x}\,Y_i(s) + x\Phi_i(s) + \sqrt{1-x}\,Y_i'(s) + (1-x)\Phi_i'(s), \\
\Pi_x(s) &:= \frac{e^{H_x(s)}p(s)}{\sum_{s'} e^{H_x(s')}},
\end{aligned}$$

where for all $i \le N$:

$$\Phi_i(s) := \sum_j c_{ij}\Phi_{\kappa_j}(s), \qquad \mathbb{E}Y_i(s)Y_i(s') = \hat{\Gamma}_i(s,s') := \sum_j c_{ij}\Gamma_{\kappa_j}(s,s'),$$
$$\Phi_i'(s) := \sum_j c_{ij}\Phi_{\kappa_j'}(s), \quad \mathbb{E}Y_i'(s)Y_i'(s') = \hat{\Gamma}_i'(s,s') := \sum_j c_{ij}\Gamma_{\kappa_j'}(s,s'),$$
$$\hat{\gamma}_i(s) := \hat{\Gamma}_i(s,s) \qquad\qquad \hat{\gamma}_i'(s) := \hat{\Gamma}_i'(s,s),$$

and let $\pi_i'(s) := \sum_{s'} \kappa_i'(s,s')$. Let $\varphi_x(s,s') := \mathbb{E}\Pi_x(s)\Pi_x(s')$. As usual, we differentiate this quantity w.r.t. x:

$$\begin{aligned}\varphi_x'(s,s') &= \mathbb{E}\Pi_x(s)\,\Pi_x(s')\Big[H_x'(s) + H_x'(s') - 2H_x'(\Pi_x)\Big]\\ &= \mathbb{E}\Pi_x(s)\,\Pi_x(s')\,\sum_t \Pi_x(t)\Big[H_x'(s) + H_x'(s') - 2H_x'(t)\Big] \end{aligned}\tag{5.8}$$

Now:

$$H_x'(s) = \frac{1}{2\sqrt{x}}Y_i(s) - \frac{1}{2\sqrt{1-x}}Y_i'(s) + \Phi_i(s) - \Phi_i'(s) \tag{5.9}$$

The last terms are handled by the following:

$$\begin{aligned}|\Phi_i(s) - \Phi_i'(s)| &= \Big|\sum_j c_{ij}\big[\gamma(s,\pi_j) - \Gamma^\star(s,s;\kappa_j) - \gamma(s,\pi_j') + \Gamma^\star(s,s;\kappa_j')\big]\Big|\\ &= \Big|\sum_j c_{ij}\big[\sum_{t,t'}\gamma(s,t)\big(\kappa_j(t,t') - \kappa_j'(t,t')\big)\\ &\qquad - \sum_{t,t'}\Gamma(s,t,s,t')\big(\kappa_j(t,t') - \kappa_j'(t,t')\big)\big]\Big|\\ &\le D_i \cdot G(s),\end{aligned}$$

where:

$$G(s) := \sum_{t,t'}\big(\gamma(s,t) + |\Gamma(s,t,s,t')|\big), \qquad \bar{G} := \max_s G(s).$$

For the first two summands of $H_x'(s)$ in (5.9) inserted at the appropriate position in (5.8), we again use partial integration and obtain:

$$\begin{aligned}&\mathbb{E}Y_i(s)\cdot\Pi_x(s)\Pi_x(s')\Pi_x(t)\\ &\quad = \tfrac{1}{2\sqrt{x}}\cdot\mathbb{E}\Pi_x(s)\Pi_x(s')\Pi_x(t)\cdot\big[\hat{\gamma}_i(s) + \hat{\Gamma}_i(s,s') + \hat{\Gamma}_i(s,t) - 3\hat{\Gamma}_i(s,\Pi_x)\big],\end{aligned}$$

and a similar expression for $Y_i'(s)$. Collecting expressions, letting:

$$V_i(s) := \hat{\gamma}_i(s) - \hat{\gamma}_i'(s), \qquad V_i(s,s') := \hat{\Gamma}_i(s,s') - \hat{\Gamma}_i'(s,s')$$

Implementing this into (5.8), this yields:

$$\frac{1}{2}\mathbb{E}\Big|\Pi_x(s)\Pi_x(s')\sum_{t,t'}\Pi_x(t)\Pi_x(t')\Big[V_i(s) + V_i(s') - 2V_i(t)$$
$$+ 2V_i(s,s') - 2V_i(s,t) - 2V_i(s',t) - 3V_i(s,t') - 3V_i(s',t') + 6V_i(t,t')\Big]\Big|$$

Now, because

$$\begin{aligned}|V_i(s,s')| &= |\hat{\Gamma}_i(s,s') - \hat{\Gamma}_i'(s,s')| \\ &= \left|\sum_j c_{ij}\sum_{t,t'}\Gamma(s,s',t,t')\cdot\big(\kappa_j(t,t') - \kappa_j'(t,t')\big)\right| \le K\cdot D_i,\end{aligned}$$

we obtain:

$$\begin{aligned}|\varphi_x'(s,s')| &= \left|\mathbb{E}\frac{d\Pi_x(s)\Pi_x(t)}{dx}\right| \le \big(11\cdot K + 4\cdot\bar{G}\big)\cdot D_i \\ &\le L\cdot\|\mathbf{C}\|_1\cdot\max_{i,s,s'}|\kappa_i(s,s') - \kappa_i'(s,s')|. \end{aligned}\tag{5.10}$$

Hence, if $\|\mathbf{C}\|_1 < \frac{1}{L}$ with $L := \big(11\cdot K + 4\cdot\bar{G}\big) \le 19K$, then the function has Lipschitz constant less than 1 w.r.t. the maximum norm, since K and $\bar{G}$ only depend on Γ. Hence, the lemma is a consequence of the Banach fixed-point theorem. $\square$

6. The Local Field and the TAP Equations

In the high enough temperature regime, as we have seen, the asymptotic independence is the main theme. Therefore, the next step in understanding the Gibbs measure is to investigate the marginal distribution of the spins. We already saw in Section 1.2.3 a way to look at this. Here, we elaborate on this topic.

The first step is the Cavity method: Fix the marginal distributions μ_i^- in the H_{N-1}-system, and observe how those marginals influence σ_N when joining the party. This gives the marginal distribution μ_N as a function of the μ_i^- in the smaller system.

But of course the advent of σ_N changes the behavior of the other σ_i. Hence, the second step then is to rewrite the Gibbs marginal distribution μ_N as a function of the other μ_i in the full system. This gives the TAP equations [19] called after Thouless, Anderson, and Palmer.

Here, we see that the asymptotic independence comes in handy. Because then the μ_i do not change too much after the return of σ_N. Now, the asymptotic independence was only looked at for a finite set of spins. But the **Local Field**, that is the part of the Hamiltonian that pertains to σ_N:

$$l_N := \sum_{i<N} \left[\sqrt{a_{iN}} g_{iN}(\sigma_i, \sigma_N) + \sqrt{b_{iN}} g_N^{(i)}(\sigma_N) + b_{iN} \Phi_{\kappa_i}(\sigma_N)\right],$$

of course depends on all the other spins, not only on the first n. Therefore, we need a better tool than Theorem 4.6, and we will start by proving a kind of central limit theorem 6.1 for this. Remarkably, both are a consequence of Theorem 4.4.

The methods of the proofs stem from Talagrand's book [16, Section 1.7]. However, in our situation, we have to give an augmented version of the Central Limit Theorem 1.7.11 there. This is, because in our case, the local field at σ_N

is not just a multiple $X \cdot \sigma_N$, but a particular Gaussian field for each value of σ_N. And by the time we use this theorem, we will have to use two potentially different assignments of σ_i, that is, two potentially different local fields. This is why we have to prove the central limit theorem for two joint Gaussian fields.

For Lemma 6.3 and Theorem 6.4, we introduce a notation for the Gibbs elementary probability of a single spin system Σ:

$$\underset{t\,/\,\Sigma}{\mathrm{ph}}\big(F(t)\big) := \frac{p(t)\cdot e^{F(t)}}{\sum_{t'\in\Sigma} p(t')\cdot e^{F(t')}}, \qquad t \in \Sigma. \tag{6.1}$$

This is some abuse of notation in that t is used both as a free parameter and a bound variable. In fact, we use this to generalize the notion of $\mathrm{th}(\cdot)$ in the Ising spin case:

$$\mathrm{th}(x) = \frac{e^x}{\mathrm{ch}(x)} - \frac{e^{-x}}{\mathrm{ch}(x)} = \sum_{t=\pm 1} t \cdot \underset{t\,/\,\{\pm 1\}}{\mathrm{ph}}(t\cdot x).$$

Looking back to (4.2) we just have

$$\Pi_i(s) = \underset{s\,/\,\Sigma}{\mathrm{ph}}\Big(Y_i(s) + \sum_j c_{ij}\Phi_{\kappa_j}(s)\Big).$$

In this whole section, we will assume for the sake of brevity $b_{iN} = 0$, $1 \le i < N$. That is, we will prove all the statements for this setting. But in the back office we will use Talagrand's class of Hamiltonians again! This is quite natural, because, again, we will compare two similar systems.

6.1. Central Limit Theorem

For this section, fix three matrices $\Gamma_1(s,s')$, $\Gamma_{12}(s,s')$, and $\Gamma_2(s,s')$, $s, s' \in \Sigma$, s.t.

$$\left(\begin{array}{c:c} \Gamma_1 & \Gamma_{12}^T \\ \hdashline \Gamma_{12} & \Gamma_2 \end{array}\right)$$

is a positive definite matrix and consider the corresponding Gaussian random field $\big(y(s), y'(s)\big)_{s\in\Sigma}$. This means we have:

$$\Gamma_1(s,s') = \mathbb{E}y(s)y(s'), \quad \Gamma_{12}(s,s') = \mathbb{E}y(s)y'(s'), \quad \Gamma_2(s,s') = \mathbb{E}y'(s)y'(s').$$

Let $\big(y_i(s), y_i'(s)\big)_{s\in\Sigma}$, $i \le N$, be an i.i.d. sequence of copies of (y, y'), independent of the g_{ij}. As always, we use the abbreviation $\gamma_\bullet(s) := \Gamma_\bullet(s, s')$ in an analogue fashion for all those matrices. We define for all $i \le N$:

$$\dot{y}_i(s) := y_i(s) - y_i(\mu_i), \qquad \dot{y}_i'(s) := y_i'(s) - y_i'(\mu_i).$$

Note that $\langle\, \dot{y}_i(\sigma_i)\,\rangle = \langle\, \dot{y}_i'(\sigma_i)\,\rangle = 0$. Let a_i, $i \le N$, be non-negative real numbers. Then, the Gaussians $\dot{S}_1 := \sum_i \sqrt{a_i}\cdot \dot{y}_i(\sigma_i)$ and $\dot{S}_2 := \sum_i \sqrt{a_i}\cdot \dot{y}_i'(\sigma_i)$ have covariances:

$$\begin{aligned}\Big\langle\, \mathbb{E}_{\mathbf{y},\mathbf{y}'} \dot{S}_1 \dot{S}_2 \,\Big\rangle &= \Big\langle\, \sum_i a_i \cdot \big[\Gamma_{12}(\sigma_i, \sigma_i) - 2\Gamma_{12}(\sigma_i, \mu_i) + \Gamma_{12}(\mu_i, \mu_i)\big] \,\Big\rangle \\ &= \sum_i a_i \cdot \big[\,\langle\, \Gamma_{12}(\sigma_i, \sigma_i)\,\rangle - \Gamma_{12}(\mu_i, \mu_i)\big]\end{aligned}$$

and corresponding expressions for the variances $\Big\langle\, \mathbb{E}\dot{S}_1^2 \,\Big\rangle$ and $\Big\langle\, \mathbb{E}\dot{S}_2^2 \,\Big\rangle$. Taking the expectation in the g_{ij}, and approximating $\pi_i(s) \approx \mathbb{E}\mu_i(s)$ and $\kappa_i(s, s') \approx \mathbb{E}\mu_i(s)\mu_i(s')$, this motivates the definition of the following covariance matrix:

$$\begin{aligned}\Phi_\bullet &:= \sum_i a_i \Big(\gamma_\bullet(\pi_i) - \Gamma_\bullet(\kappa_i)\Big) \\ &:= \sum_i a_i \Big(\sum_s \Gamma_\bullet(s, s)\pi_i(s) - \sum_{s,s'} \Gamma_\bullet(s, s')\kappa_i(s, s')\Big).\end{aligned}$$

Write $\Phi_{11} := \Phi_1$ and $\Phi_{22} := \Phi_2$. Remark that this is a function of the $\Gamma_\bullet$.

We will write $O(\mathbf{a})$ for expressions which are bounded by any linear combination of the summands in the right hand side of Theorem 4.4 times a constant which does not depend on A, i.e. N.

Theorem 6.1. *Let U be a function $\mathbb{R}^2 \to \mathbb{R}$, which is twice partial differentiable, and such that there exists a constant L with*

$$\mathbb{E}_{z_1,z_2} U(z_1, z_2)^4 < L \qquad \text{and} \qquad \mathbb{E}_{z_1,z_2}\Big(\frac{\partial^2 U(z_1,z_2)}{\partial z_1^k z_2^{2-k}}\Big)^4 < L \tag{6.2}$$

for all $0 \le k \le 2$, and for all joint Gaussian variables z_1, z_2 with variances at most $\sum_i a_i \cdot \max_{s\in\Sigma,k=1,2} |\gamma_k(s)|$.

Let ξ_1, ξ_2 be two jointly Gaussian variables with covariance matrix $\mathbb{E}\xi_i\xi_j = \Phi_{ij}$. Then, for any $H \in \mathcal{H}$:

$$\mathbb{E}\Bigg(\Bigg\langle U\left(\sum_i \sqrt{a_i}\dot{y}_i(\sigma_i), \sum_i \sqrt{a_i}\dot{y}_i'(\sigma_i)\right) - \mathbb{E}_{\xi,\xi'}U(\xi,\xi') \Bigg\rangle_H^2 \Bigg) \leq O(\mathbf{a})$$

We will use in (6.5) the additional variable of $U(x,y)$.

Proof. We use the smart path method to interpolate between the two Gaussians $\dot{S}_1^\ell$ and ξ_1 and between $\dot{S}_2^\ell$ and ξ_2. Begin by stipulating:

$$\dot{S}_1^\ell := \sum_i \sqrt{a_i}\dot{y}_i(\sigma_i^\ell) \qquad \text{and} \qquad \dot{S}_2^\ell := \sum_i \sqrt{a_i}\dot{y}_i'(\sigma_i^\ell)$$

Set $V(x,y) := U(x,y) - \mathbb{E}_{\xi_1,\xi_2}U(\xi_1,\xi_2)$. This gives $\mathbb{E}_{\xi_1,\xi_2}V(\xi_1,\xi_2) = 0$. We use replicas to calculate the following square:

$$\Big\langle V(\dot{S}_1, \dot{S}_2) \Big\rangle^2 = \Big\langle V(\dot{S}_1^1, \dot{S}_2^1)V(\dot{S}_1^2, \dot{S}_2^2) \Big\rangle$$

Since this is the left hand side of the inequality of the theorem, we will prove that the expectation of this is $O(\mathbf{a})$. The interpolation we will perform is done using independent copies $(\xi_i^1)_{i=1,2}, (\xi_i^2)_{i=1,2}$ of $(\xi_i)_{i=1,2}$ and:

$$\begin{aligned} X_i^\ell(t) &:= \sqrt{t}\dot{S}_i^\ell + \sqrt{(1-t)}\xi_i^\ell, \\ \varphi(t) &:= \mathbb{E}_{\xi_1,\xi_2}\nu_H[V(X_1^1(t), X_2^1(t)) \cdot V(X_1^2(t), X_2^2(t))]. \end{aligned}$$

Obviously, $\varphi(0) = 0$, as then $X_i^1(0)$ and $X_i^2(0)$ are independent conditioned on the disorder, and we have $\mathbb{E}_{\xi_1,\xi_2}V(\xi_1^\ell, \xi_2^\ell) = 0$.

First, we have:

$$\begin{aligned} D_{11} &:= \mathbb{E}_y \dot{S}_1^\ell \dot{S}_1^\ell = \sum_i a_i \Big(\gamma_1(\sigma_i^l) - 2\Gamma_1(\sigma_i^l, \mu_i) + \Gamma_1(\mu_i, \mu_i) \Big), \\ D_{12} &:= \mathbb{E}_{y,y'} \dot{S}_1^\ell \dot{S}_2^\ell = \sum_i a_i \Big(\gamma_{12}(\sigma_i^l) - 2\Gamma_{12}(\sigma_i^l, \mu_i) + \Gamma_{12}(\mu_i, \mu_i) \Big), \\ D_{22} &:= \mathbb{E}_{y'} \dot{S}_2^\ell \dot{S}_2^\ell = \sum_i a_i \Big(\gamma_2(\sigma_i^l) - 2\Gamma_2(\sigma_i^l, \mu_i) + \Gamma_2(\mu_i, \mu_i) \Big) \end{aligned}$$

Write

$$V^{(k_1,k_2,l_1,l_2)} := \frac{\partial^{k_1+k_2} V}{\partial x^{k_1} \partial y^{k_2}} \big(X_1^1(t), X_2^1(t)\big) \cdot \frac{\partial^{l_1+l_2} V}{\partial x^{l_1} \partial y^{l_2}} \big(X_1^2(t), X_2^2(t)\big)$$

and calculate using integration by parts

$$\nu_H[V^{(k_1,k_2,l_1,l_2)} \cdot \frac{dX_i^\ell(t)}{dt}] = \nu_H[V^{(k_1,k_2,l_1,l_2)} \cdot \left(\frac{1}{2\sqrt{t}} \dot{S}_i^\ell - \frac{1}{2\sqrt{1-t}} \xi_i^\ell\right)]$$
$$= \nu_H \left[\frac{1}{2\sqrt{t}} \sum_{j=1}^{2} D_{ij} \frac{\partial V^{(k_1,k_2,l_1,l_2)}}{\partial \dot{S}_j^\ell} - \frac{1}{2\sqrt{1-t}} \sum_{j=1}^{2} \Phi_{ij} \frac{\partial V^{(k_1,k_2,l_1,l_2)}}{\partial \xi_j^\ell} \right]$$

Then we obtain:

$$\begin{aligned}
\varphi'(t) &= \nu_H \Big[V^{(1,0,0,0)} \cdot X_1^1(t)' + V^{(0,1,0,0)} \cdot X_2^1(t)' \\
&\qquad + V^{(0,0,1,0)} \cdot X_1^2(t)' + V^{(0,0,0,1)} \cdot X_2^2(t)' \Big] \\
&= \frac{1}{2} \nu_H \Big[T^{11} V^{(2,0,0,0)} + 2T^{12} V^{(1,1,0,0)} + T^{22} V^{(0,2,0,0)} \\
&\qquad + T^{11} V^{(0,0,2,0)} + 2T^{12} V^{(0,0,1,1)} + T^{22} V^{(0,0,0,2)} \Big],
\end{aligned}$$

where $T^{ll'} := D_{ll'} - \Phi_{ll'}$ for $l, l' = 1, 2$, $l \leq l'$:

$$\begin{aligned}
T^{11} &= \sum_i a_i \Big(\gamma_1(\sigma_i^1) - \gamma_1(\pi_i) - 2\Gamma_1(\sigma_i^1, \mu_i) + \Gamma_1(\mu_i, \mu_i) + \Gamma_1(\kappa_i) \Big), \\
T^{12} &= \sum_i a_i \Big(\gamma_{12}(\sigma_i^1) - \gamma_{12}(\pi_i) - \Gamma_{12}(\sigma_i^1, \mu_i) - \Gamma_{12}(\sigma_i^2, \mu_i) \\
&\qquad + \Gamma_{12}(\mu_i, \mu_i) + \Gamma_{12}(\kappa_i) \Big), \\
T^{22} &= \sum_i a_i \Big(\gamma_2(\sigma_i^2) - \gamma_2(\pi_i) - 2\Gamma_2(\sigma_i^2, \mu_i) + \Gamma_2(\mu_i, \mu_i) + \Gamma_2(\kappa_i) \Big)
\end{aligned}$$

Therefore, we have by Cauchy-Schwarz:

$$\begin{aligned}
\varphi(1) \leq \frac{1}{2}\nu_H\Big[&T^{11}V^{(2,0,0,0)} + 2T^{12}V^{(1,1,0,0)} + T^{22}V^{(0,2,0,0)} \\
&+ T^{11}V^{(0,0,2,0)} + 2T^{12}V^{(0,0,1,1)} + T^{22}V^{(0,0,0,2)}\Big] \\
\leq L\bigg(&\sqrt{\nu_h\big[(T^{11})^2\big]} + 2\sqrt{\nu_h\big[(T^{12})^2\big]} + \sqrt{\nu_h\big[(T^{22})^2\big]}\bigg),
\end{aligned}$$

as by (6.2) we have $\nu_H\big[(V^{(k_1,k_2,l_1,l_2)})^2\big] < L$ for all the k_1, k_2, l_1, l_2 we use. Therefore, consider the second moment $\nu_H\big[(T^{11})^2\big]$:

$$\begin{aligned}
&\mathbb{E}\left\langle \left(\sum_i a_i\big(\gamma_1(\sigma_i^1) - \gamma_1(\pi_i) + \Gamma(\mu_i,\mu_i) + \Gamma_1(\kappa_i) - 2\Gamma_1(\sigma_i^1,\mu_i)\big)\right)^2 \right\rangle \\
&= \mathbb{E}\bigg\langle\bigg(\sum_s \gamma_1(s)\sum_i a_i\big[\delta_i^1(s) - \pi_i(s)\big] \\
&\qquad + \sum_{s,s'}\Gamma_1(s,s')\sum_i a_i\big[\kappa_i(s,s') + \mu_i(s)\mu_i(s') - 2\delta_i^1(s)\mu_i(s')\big]\bigg)^2\bigg\rangle
\end{aligned}$$

which can be bound by:

$$\begin{aligned}
&2\,\mathbb{E}\left\langle \left(\sum_s \gamma_1(s)\sum_i a_i\big[\delta_i^1(s) - \pi_i(s)\big]\right)^2 \right\rangle \\
&\quad + 2\,\mathbb{E}\left\langle \left(\sum_{s,s'}\Gamma_1(s,s')\sum_i a_i\big[\kappa_i(s,s') + \mu_i(s)\mu_i(s') - 2\delta_i^1(s)\mu_i(s')\big]\right)^2 \right\rangle
\end{aligned}$$

The first term is handled by Theorem 4.4. The second term is handled in the following way:

$$\mathbb{E}\left\langle \left(\sum_{s,s'}\Gamma_1(s,s')\sum_i a_i\big[\kappa_i(s,s')+\mu_i(s)\mu_i(s')-2\delta_i^1(s)\mu_i(s')\big]\right)^2 \right\rangle$$
$$\leq K\sum_{s,s'}\mathbb{E}\left\langle \left(\sum_i a_i\big[\kappa_i(s,s')+\mu_i(s)\mu_i(s')-2\delta_i^1(s)\mu_i(s')\big]\right)^2 \right\rangle$$

By Jensen's inequality this can be bound by:

$$K\sum_{s,s'}\mathbb{E}\left\langle \left(\sum_i a_i\big[\kappa_i(s,s')+\delta_i^{3,4}(s,s')-2\delta_i^{1,2}(s,s')\big]\right)^2 \right\rangle$$
$$= K\sum_{s,s'}\mathbb{E}\left\langle \left(\sum_i a_i\big[\delta_i^{3,4}(s,s')-\kappa_i(s,s')+2\kappa_i(s,s')-2\delta_i^{1,2}(s,s')\big]\right)^2 \right\rangle$$
$$\leq 10K\sum_{s,s'}\mathbb{E}\left\langle \left(\sum_i a_i\big[\delta_i^{1,2}(s,s')-\kappa_i(s,s')\big]\right)^2 \right\rangle$$

T^{12} and T^{22} are handled in the same way. □

6.2. Cavity Equation

The very idea of the cavity method would be to write

$$\langle\, f(\sigma_N)\,\rangle = \frac{\mathrm{Tr}_t\, f(t)\cdot\langle\, \mathcal{E}(t,\boldsymbol{\sigma})\,\rangle_-}{\mathrm{Tr}_t\,\langle\, \mathcal{E}(t,\boldsymbol{\sigma})\,\rangle_-}, \qquad \mathcal{E}(t,\boldsymbol{\sigma}) = e^{\sum_i \sqrt{a_{iN}}\, g_{iN}(\sigma_i,t)},$$

where $\langle\,\cdot\,\rangle_-$ would be just the Gibbs measure where all the a_{iN} are set to zero, leaving σ_N independent of the other $\sigma_1,\dots,\sigma_{N-1}$. Therefore, σ_N now has the role of a newcomer.

But in our case, this would lead to an intractable situation, because we need to let the π_i and κ_i be invariant. Therefore, we need to stay in $\mathcal{H}$ and the Gaussian fields have to be compensated in the exponent. Hence, given the Hamiltonian $H\in\mathcal{H}$, we work with the changed Hamiltonian H^-, where all the $a_{i,N}$, $i=1,\dots,N-1$ are replaced with 0 and the b_{iN} are replaced with a_{iN}. This then gives us both $H, H^-\in\mathcal{H}$. On the other hand, this then introduces the compensatory Gaussians $g_i^{(N)}(\cdot)$ and deterministic fields $\Phi_i^{(N)}(\cdot)$. Hence the definition of $\mathcal{E}(\boldsymbol{\sigma},t)$ is becoming more complicated than we stated above:

Lemma 6.2 (Cavity Equation)**.** *Let $H\in\mathcal{H}$. Suppose all $b_{ij}=0$. Let $H^-\in\mathcal{H}$ be s.t. all direct interaction a_{iN} is shifted to the $b_{iN}=b_{Ni}$. Then, we have for any function $f\colon \Sigma^{N-1}\times\Sigma\to\mathbb{R}$, using $\boldsymbol{\rho} := (\sigma_1,\dots,\sigma_{N-1})$:*

$$\langle\, f(\boldsymbol{\rho},\sigma_N)\,\rangle_H = \frac{\mathrm{Tr}_t\, f(\boldsymbol{\rho},t)\cdot\langle\, \mathcal{E}(t,\boldsymbol{\sigma})\,\rangle_{H^-}}{\mathrm{Tr}_t\,\langle\, \mathcal{E}(t,\boldsymbol{\sigma})\,\rangle_{H^-}},$$
$$\mathcal{E}(t,\boldsymbol{\sigma}) := \exp\Big[\sum_{i<N}\sqrt{a_{iN}}\big(g_{iN}(\sigma_i,t) - g_i^{(N)}(\sigma_i)\big) - \sum_i a_{iN}\Phi_{\kappa_N}(\sigma_i)\Big].$$

Proof. By definition, we have:

$$\begin{aligned}\langle\, f(\boldsymbol{\rho},\sigma_N)\,\rangle_H &= \frac{1}{Z_N}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\, f(\boldsymbol{\rho},\sigma_N)\exp\Big[\sum_{i<j}\sqrt{a_{ij}}\,g_{ij}(\sigma_i,\sigma_j)\Big]\\ &= \frac{1}{Z_N}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\,\mathrm{Tr}_t\, f(\boldsymbol{\rho},t)\exp\Big[\sum_{i<N}\sqrt{a_{iN}}\,g_{iN}(\sigma_i,t)\Big]\\ &\qquad\cdot\exp\Big[\sum_{i<j<N}\sqrt{a_{ij}}\,g_{ij}(\sigma_i,\sigma_j)\Big]\end{aligned}$$

Here, we replaced the average in the last spin σ_N by an average in t, although in $\mathrm{Tr}_{\boldsymbol{\sigma}}\cdot$ there still is the average over all $\boldsymbol{\sigma} \in \Sigma^N$ – in particular over σ_N as well, even though the function does not evaluate σ_N here anymore. Remark that we have:

$$\begin{aligned} H^-(\boldsymbol{\sigma}) = &\sum_{i<j<N} \sqrt{a_{ij}} g_{ij}(\sigma_i, \sigma_j) + \sum_{i<N} \sqrt{a_{iN}} \big(g_i^{(N)}(\sigma_i) + g_N^{(i)}(\sigma_N) \big) \\ &+ \sum_i a_{iN} \big(\Phi_{\kappa_N}(\sigma_i) + \Phi_{\kappa_i}(\sigma_N) \big) \end{aligned}$$

Therefore, we can rewrite $\langle\, f(\boldsymbol{\rho}, \sigma_N)\, \rangle_H$:

$$\begin{aligned} &\frac{1}{Z_N} \mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{H^-(\boldsymbol{\sigma})}\, \mathrm{Tr}_t\, f(\boldsymbol{\rho}, t)\, e^{\sum_{i<N} \sqrt{a_{iN}} \left(g_{iN}(\sigma_i, t) - g_i^{(N)}(\sigma_i) - g_N^{(i)}(\sigma_N) \right)} \\ &\qquad\qquad \cdot e^{-\sum_i a_{iN} \left(\Phi_{\kappa_N}(\sigma_i) + \Phi_{\kappa_i}(\sigma_N) \right)} \\ &= \frac{1}{Z'} \Big\langle\, \mathrm{Tr}_t\, f(\boldsymbol{\rho}, t)\, e^{\sum_{i<N} \sqrt{a_{iN}} \left(g_{iN}(\sigma_i, t) - g_i^{(N)}(\sigma_i) - g_N^{(i)}(\sigma_N) \right)} \\ &\qquad\qquad \cdot e^{-\sum_i a_{iN} \left(\Phi_{\kappa_N}(\sigma_i) + \Phi_{\kappa_i}(\sigma_N) \right)} \Big\rangle_{H^-} \\ &= \frac{1}{Z'} \Big\langle\, \mathrm{Tr}_t\, f(\boldsymbol{\rho}, t)\, e^{\sum_{i<N} \sqrt{a_{iN}} \left(g_{iN}(\sigma_i, t) - g_i^{(N)}(\sigma_i) \right) - \sum_i a_{iN} \Phi_{\kappa_N}(\sigma_i)} \\ &\qquad\qquad \cdot e^{-\sum_{i<N} \sqrt{a_{iN}} g_N^{(i)}(\sigma_N) - \sum_i a_{iN} \Phi_{\kappa_i}(\sigma_N)} \Big\rangle_{H^-}. \end{aligned}$$

Here, Z' is the appropriate normalizer, i.e. the remainder of the expression evaluated at $f \equiv 1$. Now, under $\langle \cdot \rangle_{H_-}$, the σ_N is independent of the other spins, hence:

$$\begin{aligned} \langle\, f(\boldsymbol{\rho}, \sigma_N)\, \rangle_H = &\frac{1}{Z'} \Big\langle\, e^{-\sum_{i<N} \sqrt{a_{iN}} g_N^{(i)}(\sigma_N) - \sum_i a_{iN} \Phi_{\kappa_i}(\sigma_N)} \Big\rangle_{H^-} \\ &\cdot \Big\langle\, \mathrm{Tr}_t\, f(\boldsymbol{\rho}, t)\, e^{\sum_{i<N} \sqrt{a_{iN}} \left(g_{iN}(\sigma_i, t) - g_i^{(N)}(\sigma_i) \right) - \sum_i a_{iN} \Phi_{\kappa_N}(\sigma_i)} \Big\rangle_{H^-} \end{aligned}$$

The first factor cancels out with the normalizing factor, which finishes the proof. □

Before we get into the TAP equations, we need some calculations. In the lemma above, we changed the dependence on σ_N to the dependence on t.

Thus, we now have a newcomer t and need to give all interactions with it. But there is a catch: we have to incorporate the $g_i^{(N)}(\cdot)$ into this interaction, as will become clear in the proof of Theorem 6.4. Therefore, we fix a symmetric positive semidefinite matrix $\tilde{\Gamma}(s,t,s',t')$, $s,t,s',t' \in \Sigma$, though we do not assume $\tilde{\Gamma}^\star$ to be symmetric, in contrast to what we usually do for all our $\Gamma^\star$. Therefore, given $\tilde{\Gamma}$, we consider the corresponding i.i.d. sequence of Gaussian fields $\big(g_i(s,t)\big)_{s,t}$. Further, fix some $\tilde{\pi}(\cdot)$ and $\tilde{\kappa}(\cdot,\cdot)$ and let $a_1,\dots,a_N \geq 0$. Then, define for $t,t' \in \Sigma$ the deterministic fields:

$$\tilde{\Phi}^{(i)}(t) := \tfrac{1}{2}\big(\tilde{\gamma}(\pi_i,t) - \tilde{\Gamma}^\star(\kappa_i;t,t)\big), \qquad \hat{\Phi}(s) := \tfrac{1}{2}\big(\gamma(s,\tilde{\pi}) - \Gamma^\star(s,s;\tilde{\kappa})\big).$$

Next, we give the appropriate version of the $\mathcal{E}(t,\boldsymbol{\sigma})$ in the above lemma by writing for all $t \in \Sigma$ and $\boldsymbol{\sigma} \in \Sigma^N$:

$$\tilde{\mathcal{E}}(t,\boldsymbol{\sigma}) := \exp\left[\sum_i \sqrt{a_i}\, g_i(\sigma_i,t) - \sum_i a_i \hat{\Phi}(\sigma_i)\right].$$

The same term, but given for the mean of the exponent, is for $t \in \Sigma$:

$$\bar{\mathcal{E}}(t) := \exp\left[\sum_i \sqrt{a_i}\, g_i(\mu_i,t) - \sum_i a_i \hat{\Phi}(\pi_i)\right]$$

Finally, we introduce a term that will become the so-called Onsager term in the setting of the TAP equations:

$$\tilde{\Psi}(t,t') := \sum_i a_i \big[\sum_s \pi_i(s)\tilde{\Gamma}(s,t,s,t') - \tilde{\Gamma}^\star(t,t';\kappa_i)\big].$$

Equipped with Theorem 6.1, we get the following Lemma, recalling (6.1):

Lemma 6.3. *For every $H \in \mathcal{H}$ and all $t \in \Sigma$:*

$$\mathbb{E}\Bigg| \exp\left[\sum_i \sqrt{a_i} g_i(\mu_i, t) + \sum_i a_i \tilde{\Phi}^{(i)}(t) - \sum_i a_i \hat{\Phi}(\pi_i)\right] \\ - \Big\langle\, \tilde{\mathcal{E}}(t, \boldsymbol{\sigma}) \,\Big\rangle_H \Bigg| \le O(\mathbf{a}), \tag{6.3}$$

$$\mathbb{E}\Bigg| \operatorname*{ph}_{t\,/\,\Sigma}\left[\sum_i \sqrt{a_i} g_i(\mu_i, t) + \sum_i a_i \tilde{\Phi}^{(i)}(t) - \sum_i a_i \hat{\Phi}(\pi_i)\right] \\ - \frac{\Big\langle\, p(t)\tilde{\mathcal{E}}(t, \boldsymbol{\sigma}) \,\Big\rangle_H}{\Big\langle\, \mathrm{Tr}_{t'}\, \tilde{\mathcal{E}}(t', \boldsymbol{\sigma}) \,\Big\rangle_H} \Bigg| \le O(\mathbf{a}), \tag{6.4}$$

$$\mathbb{E}\Bigg| \Big\langle\, \mathrm{Tr}_{t'} \sum_i \sqrt{a_i}\big(g_i(\sigma_i, t) - g_i(\mu_i, t)\big)\tilde{\mathcal{E}}(t', \boldsymbol{\sigma}) \,\Big\rangle_H \\ - \mathrm{Tr}_{t'}\, \tilde{\Psi}(t, t') e^{\sum_i a_i \tilde{\Phi}^{(i)}(t')} \bar{\mathcal{E}}(t') \Bigg| \le O(\mathbf{a}). \tag{6.5}$$

Actually, the last summand in the $\mathrm{ph}_{t\,/\,\Sigma}[\cdot]$ term of (6.4) does not depend on t, and hence cancels out without changing the expression.

Proof. Fix $t \in \Sigma$ and $H \in \mathcal{H}$ and let $\langle\, \cdot \,\rangle := \langle\, \cdot \,\rangle_H$.
Inequality (6.3) will be proved using Theorem 6.1 for the function $(x, y) \mapsto e^x$ and $y_i(s) := g_i(s, t)$, which means $\Gamma_1(s, s') = \tilde{\Gamma}(s, t, s', t)$. Then, in the theorem this yields $\Phi_1 = 2\sum_i a_i \tilde{\Phi}^{(i)}(t)$.
But first, in order to prove inequality (6.3), we calculate for its left hand side:

$$\mathbb{E}_{g_1,\ldots,g_N}\Big|\Big\langle e^{\sum_i \sqrt{a_i} y_i(\sigma_i) - \sum_i a_i \hat{\Phi}(\sigma_i)} - e^{\sum_i \sqrt{a_i} y_i(\mu_i) + \frac{\Phi_1}{2} - \sum_i a_i \hat{\Phi}(\pi_i)}\Big\rangle\Big|$$
$$\le e^{-\sum_i a_i \hat{\Phi}(\pi_i)} \cdot \mathbb{E}_{g_1,\ldots,g_N}\Big|\Big\langle e^{\sum_i \sqrt{a_i} y_i(\sigma_i) - \sum_i a_i \hat{\Phi}(\delta_i - \pi_i)} - e^{\sum_i \sqrt{a_i} y_i(\mu_i) + \frac{\Phi_1}{2}}\Big\rangle\Big|$$

Now, we know

$$\left(\sum_i a_i \sum_s \Phi(s) \cdot \big(\delta_i(s) - \pi_i(s)\big)\right)^2 \le K \max_s \left(\sum_i a_i\big(\delta_i(s) - \pi_i(s)\big)\right)^2$$

to be $O(\mathbf{a})$ and exp is Lipschitz on any compact interval, say here with constant L, hence:

$$\mathbb{E}\left\langle\ e^{\sum_i a_i \hat{\Phi}(\delta_i - \pi_i)} - 1\ \right\rangle^2 \leq O(\mathbf{a}).$$

Then, using triangle and Cauchy-Schwarz inequalities, we can, by accepting an $O(\mathbf{a})$ error, delete this term in the exponent. Hence, (6.3) follows if we can bound the simpler quantity:

$$\begin{aligned}
&\mathbb{E}_{g_1,\ldots,g_N}\left|\left\langle\ e^{\sum_i \sqrt{a_i} y_i(\sigma_i)} - e^{\sum_i \sqrt{a_i} y_i(\mu_i)} \cdot e^{\frac{\Phi_1}{2}}\ \right\rangle\right| \\
&\qquad = \mathbb{E}_{g_1,\ldots,g_N}\left|e^{\sum_i \sqrt{a_i} y_i(\mu_i)} \cdot \left\langle\ e^{\sum_i \sqrt{a_i} \dot{y}_i(\sigma_i)} - e^{\frac{\Phi_1}{2}}\ \right\rangle\right| \\
&\qquad \leq \mathbb{E}_{g_1,\ldots,g_N} e^{2\sum_i \sqrt{a_i} y_i(\mu_i)} \cdot \mathbb{E}_{g_1,\ldots,g_N}\left\langle\ e^{\sum_i \sqrt{a_i} \dot{y}_i(\sigma_i)} - e^{\frac{\Phi_1}{2}}\ \right\rangle^2
\end{aligned}$$

Since $\mathbb{E}_\xi e^{\sqrt{\Phi_1}\,\xi} = e^{\frac{\Phi_1}{2}}$, Theorem 6.1 implies that this is $O(\mathbf{a})$, which proves (6.3).

Next, we obtain (6.4) from (6.3) by using the elementary inequality

$$\left|\frac{A'}{B'} - \frac{A}{B}\right| \leq |A - A'| + |B - B'|$$

which holds whenever $|A'| \leq B'$ and $B \geq 1$.

Finally, for (6.5), we again use Theorem 6.1, but this time with the function $U(x,y) = x\,e^y$ and the fields $y_i(s) := g_i(s,t)$ and $y_i'(s) := g_i(s,t')$. This yields $\Phi_1 = 2\sum_i a_i \tilde{\Phi}^{(i)}(t)$, $\Phi_2 = 2\sum_i a_i \tilde{\Phi}^{(i)}(t')$ and $\Phi_{12} = \tilde{\Psi}(t,t')$. Then, using the same calculation as in the proof of (6.3) and observing that by partial integration:

$$\mathbb{E}\xi_1 e^{\xi_2} = \mathbb{E}\xi_1\xi_2 \cdot \mathbb{E}e^{\xi_2} = \tilde{\Psi}(t,t')\,e^{\frac{1}{2}\tilde{\Phi}(t')},$$

this proves (6.5). $\square$

6.3. TAP Equations

Finally, the TAP-equations are given by:

Theorem 6.4. *Let $t \in \Sigma$. Then*

$$\mathbb{E}\left|\mu_N(t) - \underset{t\,/\,\Sigma}{\mathrm{ph}}\left[\sum_i \sqrt{a_{iN}}g_{iN}(\mu_i, t) - \Psi^{(N)}(t, \mu_N) + \sum_i a_{iN}\Phi_{\kappa_i}(t)\right]\right| \\ \leq O(\mathbf{a}),$$

where we have the generalized ***Onsager term:***

$$\Psi^{(N)}(t, t') := \sum_i a_{iN}\big(\sum_s \pi_i(s)\Gamma(s, t, s, t') - \Gamma^\star(t, t'; \kappa_i)\big).$$

Of course this is true for all μ_i by the obvious changes. In particular, we can also define $\Psi^{(i)}(s, s') := \sum_j a_{ij}\big(\sum_t \pi_j(t)\Gamma(s, t, s', t) - \Gamma_{\kappa_j}(s, s')\big)$. Observe that we have seen the Onsager term from the beginning in $\Phi^{(\mathbf{A},i)}(s) = \sum_j a_{ij}\Phi_{\kappa_j}(s) = \frac{1}{2}\Psi^{(i)}(s, s)$.

Proof. First, using Lemma 6.2, we have:

$$\mu_N(t) = \langle\, \delta_N(t)\, \rangle = \frac{\langle\, p(t)\mathcal{E}(t, \boldsymbol{\sigma})\, \rangle_-}{\mathrm{Tr}_{t'}\,\langle\, \mathcal{E}(t', \boldsymbol{\sigma})\, \rangle_-}, \tag{6.6}$$

$$\mathcal{E}(t, \boldsymbol{\sigma}) = \exp\left[\sum_{i<N} \sqrt{a_{iN}}\big(g_{iN}(\sigma_i, t) - g_i^{(N)}(\sigma_i)\big) - \sum_i a_{iN}\Phi_{\kappa_N}(\sigma_i)\right],$$

where $\langle \cdot \rangle_-$ is the Gibbs expectation w.r.t. the Hamiltonian H^-. We set $\mu_i^-(s) := \langle\, \delta_i(s)\, \rangle_-$. Under $\langle \cdot \rangle_-$, σ_N is independent of the other spins. We will apply Lemma 6.3 for the Hamiltonian H^-. Hence, all μ_i there will be replaced in the current context as μ_i^-. Further, we stipulate:

$$\tilde{g}_i(s, t) := g_{iN}(s, t) - g_i^{(N)}(s),$$

which is not symmetric anymore as we anticipated by using $\tilde{\Gamma}$. Hence, we will use the following covariance matrix for Lemma 6.3:

$$\tilde{\Gamma}(s, t, s', t') := \mathbb{E}\tilde{g}_i(s, t)\tilde{g}_i(s', t') = \Gamma(s, t, s', t') + \Gamma^\star(s, s'; \kappa_N).$$

We set $\tilde{\pi} := \pi_N$ and $\tilde{\kappa} := \kappa_N$. This implies $\hat{\Phi}(s) = \Phi_{\kappa_N}(s)$ for all s. Finally, we obtain:

$$\tilde{\Psi}(t, t') = \Psi^{(N)}(t, t') + \sum_i a_{iN}\left(\sum_s \Gamma^\star(s, s; \kappa_N)\pi_i(s) - \Gamma^\star(\kappa_i; \kappa_N)\right).$$

Remark that the second part is constant and does not depend on t, t', because $\tilde{\Gamma}^\star$ is not symmetric.

We start the calculation of $\mu_N(t)$ in (6.6) by using (6.4):

$$\mathbb{E}\Bigg| \underset{t \,/\, \Sigma}{\mathrm{ph}} \left[\sum_{i<N} \sqrt{a_{iN}}\tilde{g}_i(\mu_i^-, t) + \tfrac{1}{2}\tilde{\Psi}(t,t) - \sum_{i<N} a_{iN}\Phi_{\kappa_N}(\pi_i) \right] - \frac{\langle\, p(t)\mathcal{E}(t,\boldsymbol{\sigma})\,\rangle_-}{\langle\, \mathrm{Tr}_{t'}\, \mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-} \Bigg| \le O(\mathbf{a}) \tag{6.7}$$

Because, as in the remark before the proof, we have $\sum_{i<N} a_{iN}\tilde{\Phi}_{\kappa_i}(t) = \frac{1}{2}\tilde{\Psi}(t,t)$, for all $t \in \Sigma$. We start by handling the first summand $\tilde{g}_i(\mu_i^-, t)$. We obtain, that

$$\mathbb{E}\Bigg| \frac{\langle\, \sum_i \sqrt{a_{iN}}\big(\tilde{g}_{iN}(\sigma_i, t) - \tilde{g}_{iN}(\mu_i^-, t)\big)\,\mathrm{Tr}_{t'}\,\mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-}{\langle\, \mathrm{Tr}_{t'}\,\mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-} - \sum_{t'} \tilde{\Psi}(t,t') \underset{t' \,/\, \Sigma}{\mathrm{ph}} \left[\sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i^-, t') + \frac{1}{2}\Psi(t',t') - \sum_i a_{iN}\Phi_{\kappa_N}(\pi_i) \right] \Bigg|$$

is $O(\mathbf{a})$ by using (6.5) in the numerator and (6.3) in the denominator of the first term. Now, the first fraction is:

$$\begin{aligned}
&\frac{\langle\, \sum_i \sqrt{a_{iN}}\big(\tilde{g}_{iN}(\sigma_i, t) - \tilde{g}_{iN}(\mu_i^-, t)\big)\,\mathrm{Tr}_{t'}\,\mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-}{\langle\, \mathrm{Tr}_{t'}\,\mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-} \\
&= \frac{\langle\, \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\sigma_i, t)\,\mathrm{Tr}_{t'}\,\mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-}{\langle\, \mathrm{Tr}_{t'}\,\mathcal{E}(t',\boldsymbol{\sigma})\,\rangle_-} - \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i^-, t) \\
&= \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i, t) - \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i^-, t),
\end{aligned}$$

because by Lemma 6.2 the first term here is equal to

$$\left\langle \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\sigma_i, t) \right\rangle = \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i, t).$$

Hence, this gives us:

$$\mathbb{E}\Bigg|\sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i,t) - \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i^-,t) \\ - \sum_{t'} \tilde{\Psi}(t,t') \operatorname*{ph}_{t'\,/\,\Sigma}\Big[\sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i^-,t') + \tfrac{1}{2}\tilde{\Psi}(t',t') - \sum_i a_{iN}\Phi_{\kappa_N}(\pi_i)\Big]\Bigg| \\ \leq O(\mathbf{a})$$

Now, the $\mathrm{ph}_{t'\,/\,\Sigma}[\cdot]$ term is by (6.7) and (6.6) seen to be in $O(\mathbf{a})$ distance of μ_N, hence we have by the triangle inequality:

$$\mathbb{E}\Big|\sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i,t) - \sum_i \sqrt{a_{iN}}\tilde{g}_{iN}(\mu_i^-,t) - \sum_{t'} \tilde{\Psi}(t,t')\mu_N(t')\Big| \leq O(\mathbf{a})$$

Putting this in the argument of the $\mathrm{ph}_{t\,/\,\Sigma}[\cdot]$ in (6.7), we have proved:

$$\mathbb{E}\Bigg|\mu_N(t) - \operatorname*{ph}_{t\,/\,\Sigma}\Big[\sum_i \sqrt{a_{iN}}\big(g_{iN}(\mu_i,t) - g_i^{(N)}(\mu_i)\big) \\ + \tfrac{1}{2}\tilde{\Psi}(t,t) - \tilde{\Psi}(t,\mu_N) - \sum_i a_{iN}\Phi_{\kappa_N}(\pi_i)\Big]\Bigg| \\ \leq O(\mathbf{a}).$$

Now, observe that any summand which does not depend on t, cancels out in $\mathrm{ph}_{t\,/\,\Sigma}[\cdot]$. Therefore, we can replace the $\tilde{\Psi}$ by Ψ and drop $g_i^{(N)}(\mu_i)$ as well as the last term. Then, the $\mathrm{ph}_{t\,/\,\Sigma}$ in this inequality is equal to:

$$\operatorname*{ph}_{t\,/\,\Sigma}\Big[\sum_i \sqrt{a_{iN}}g_{iN}(\mu_i,t)s + \tfrac{1}{2}\Psi^{(N)}(t,t) - \Psi^{(N)}(t,\mu_N)\Big]$$

Finally, the theorem follows because, as we have seen,

$$\tfrac{1}{2}\Psi^{(N)}(t,t) = \sum_i a_{iN}\Phi_{\kappa_i}(t).$$

□

7. The SK Model with Gaussian Random Field Interaction on ∞ Spin Sets

Up to now we only considered finite Σ. Hence, we did not yet embrace the full d-component spin SK model. Now, we intend to remedy this. In the finite setting, we considered Gaussian random fields $g(s, t)$ on Σ^2. This is what we will still do even when Σ is infinite by use of general bounded Gaussian random fields (GRF). However, asking for boundedness has not been enough for us to prove the existence of the order parameters and the main theorems in all generality. Still, we could achieve both results for a generalization of the d-component spin SK model. For the theory of Gaussian Random Fields we refer the reader to [14] or [1].

We will start by defining the model in the more general setting and give some technical remarks and assumptions we will use later on. Next, we discuss the definition and existence of the order parameters, which seems, in contrast to the finite case, to be non-trivial – at least we were able to prove the well-definedness of the fixed point equation only under restrictions on Γ. Then, after performing as an examplary calculation the Guerra Interpolation, we proceed to generalize the proofs of Chapter 5 that were generalizations of [15]. There are some larger parts of those arguments that just carry over to the current setting and we will restate those here quickly again. But other parts need to be done in more detail – most importantly the Cauchy-Schwarz step in the proof of the main lemma needs another restriction on Γ. But it is still amazing how easy most parts of Talagrands argument carry over because they do not depend to heavily on the structure of Σ and Γ.

7.1. Definitions and Assumptions

Let $(\Sigma, \mathcal{F}, p)$ be a probability space. Again, p will play the role of the a-priori measure. Then, for any $\mathcal{F}^{\otimes N}$-measurable (m.b.) function f, we set:

$$\mathrm{Tr}_{\boldsymbol{\sigma}\in\Sigma^N} f(\boldsymbol{\sigma}) := \int_{\Sigma^N} f(\boldsymbol{\sigma}) p^{\otimes N}(d\boldsymbol{\sigma}).$$

Now, suppose we are given a centered Gaussian random field $\big(g(s,t)\big)_{s,t\in\Sigma}$ on Σ^2. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be the probability space of this Gaussian field as well as of the other fields we will introduce. Of course we denote expectation w.r.t. this $\mathbb{P}$ by $\mathbb{E}\cdot$ and the covariance function is:

$$\Gamma(s,t,s',t') := \mathbb{E}g(s,t)g(s',t')$$

Note: We will assume Γ such that the Gaussian random field has bounded sample paths $\mathbb{P}$-a.s.[1] In particular, Γ is bounded. Let

$$K_\Gamma := \sup_{s,t,s',t'} |\Gamma(s,t,s',t')|.$$

Now, define the Hamiltonian, partition function, and Gibbs measure as usual:

$$\begin{aligned} H_N(\boldsymbol{\sigma}) &= \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j), \qquad \forall \boldsymbol{\sigma} \in \Sigma^N, \\ Z_N &= \mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{H_N(\boldsymbol{\sigma})}, \\ \langle\, f(\boldsymbol{\sigma})\,\rangle &= \tfrac{1}{Z_N} \mathrm{Tr}_{\boldsymbol{\sigma}}\, f(\boldsymbol{\sigma}) \cdot e^{H_N(\boldsymbol{\sigma})}, \qquad \nu[f(\boldsymbol{\sigma})] := \mathbb{E}\,\langle\, f(\boldsymbol{\sigma})\,\rangle\,. \end{aligned}$$

Because $g(\cdot,\cdot)$ is bounded a.s. the partition function $Z_N < \infty$ a.s.

7.1.1. Technical Aspects and Assumptions

We introduce several technical notions and assumptions for the general Gaussian Random Fields and their covariance functions Γ.

[1] Actually we would need for the partition function to be finite that it has $\mathbb{P}$-a.s. all exponential moments:

$$\int_\Sigma \int_\Sigma e^{\beta g(s,t)} p(ds)p(dt) < \infty \qquad \text{for all } \beta \geq 0 \text{ and } \mathbb{P}\text{-a.s.} \tag{7.1}$$

But then later things would get much more complicated.

First we will need the following space of functions that will serve as densities w.r.t. $p^{\otimes 2}$:

Definition 7.1. *First, let:*

$$\mathcal{M}_1 := \left\{ f \in L^\infty(\Sigma^2, \mathbb{R}_{\geq 0}) \mid f(\cdot,\cdot) \text{ is symmetric and } \mathrm{Tr}_{t,t'} f(t,t') = 1 \right\}$$

We will denote by $\mathcal{M}_+$ the set of functions $\kappa \in \mathcal{M}_1$ which are positive definite.

Next, most importantly, we recall the following result on bounded GRFs due to Talagrand [14]:

Remark 7.2.
We will use Talagrand's expansion for bounded Gaussian fields. Therefore, we assume our index spaces $(\Sigma, \mathcal{F})$ and $(\Sigma^2, \mathcal{F}^{\otimes 2})$ to be such that [14] is applicable, mainly that Σ^2 should be separable in the d_Γ-metric induced by Γ: $d_\Gamma\big((s,t),(s',t')\big) = \mathbb{E}(g(s,t)-g(s',t'))^2$. This is met for instance when $\Sigma \subset \mathbb{R}^d$ is an open or closed domain, $\mathcal{F}$ is the Borel σ-algebra, and Γ is continuous near the diagonal.

Now, since $g(\cdot,\cdot)$ is assumed to be a bounded process, by Talagrand's expansion [14, Theorem 2], there are a sequence of centered Gaussians $(Y_n)_{n\in\mathbb{N}}$ with variances $O(\frac{1}{\log n})$ and a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset L^\infty(\Sigma^2, [0,1])$ s.t.:

$$g(s,t) = \sum_{n\in\mathbb{N}} \alpha_n(s,t) Y_n,$$

where the series converges a.s. and in L^2 and $\sum_n \alpha_n(s,t) \leq 1$, for all $s,t \in \Sigma$. Observe that the α_n are $\mathcal{F}^{\otimes 2}$-measurable and Y_n is a random variable. Therefore, $g(s,t)$ is $\mathcal{F}^{\otimes 2} \otimes \mathcal{G}$-measurable.

Correspondingly, considering for $f \in \mathcal{M}_+$ the Gaussian field $\big(g^{(f)}(s)\big)_{s\in\Sigma}$ with covariance function Γ_f, then it is $\mathcal{F} \otimes \mathcal{G}$-measurable if it has bounded paths.

A corollary of this remark is that we always will be able to write Γ as:

$$\Gamma(s,t,s',t') = \sum_{i,j=1}^{\infty} \alpha_i(s,t) M_{ij} \alpha_j(s',t'), \qquad s,t,s',t' \in \Sigma,$$

where M_{ij} is a symmetric positive semidefinite matrix with $M_{ii} \leq \frac{L}{\log i}$ and the $(\alpha_i)_i$ are as in the remark.

Now we turn to the two technical assumptions that will be needed for the proofs. It is not clear whether they are necessary for the formulas to be true. The first one will be used to prove the existence of the order parameters in the next section:

Assumption 7.3. *Assume that $\Sigma \subset \mathbb{R}^d$ and Γ can be written in the form:*

$$\Gamma(s,t,s',t') = \sum_{i,j=1}^{\infty} \alpha_i(s,t) M_{ij} \alpha_j(s',t'), \qquad s,t,s',t' \in \Sigma,$$

with $\alpha_i(s,t) \in [0,1]$ being symmetric positive definite functions for all i, and $\mathbf{M} = (M_{ij})_{ij}$ a symmetric positive semidefinite matrix with $M_{ii} \leq \frac{L}{\log(i)}$.

The second assumption is used for the Cauchy-Schwarz step in the main lemma proof:

Assumption 7.4. *Assume*

$$\Gamma(s,t,s',t') = \sum_{k,l=1}^{\infty} \hat{M}_{kl} r_k(s) r_k(t) r_l(s') r_l(t') \tag{7.2}$$

where $r_k(s) \in [0,1]$ for all s and all k and assume $K := \sum_{k,l} |\hat{M}_{kl}| < \infty$.

Both assumptions dwell on a diagonalizability of the α_i:

$$\alpha_i(s,t) = \sum_{k=1}^{\infty} \lambda_k^{(i)} a_k^{(i)}(s) a_k^{(i)}(t), \tag{7.3}$$

although they focus on different aspects. Of course, diagonalization is possible given if the α_i are symmetric, implying that $g(s,t)$ is symmetric a.s. Then, Assumption 7.3 is the case where the α_i are positive definite, i.e. $\lambda_k^{(i)} \geq 0$ for all $i,k \in \mathbb{N}$.

Assumption 7.4 points in a different direction as it assumes absolute summability for a matrix $\hat{\mathbf{M}}$, which we now derive from the diagonalization (7.3). Actually, we will write (7.2) using the countable index set $\mathbb{N}^2$. Let $\hat{a}_{i,k} := \|a_k^{(i)}\|_\infty$ and set

$$r_{(i,k)}(s) := \frac{1}{\hat{a}_{i,k}} a_k^{(i)}(s), \qquad \hat{M}_{(i,k),(j,l)} = M_{ij} \cdot \lambda_k^{(i)} \hat{a}_{i,k}^2 \cdot \lambda_l^{(j)} \hat{a}_{j,l}^2$$

Then this yields:

$$\Gamma(s,t,s',t') = \sum_{(i,k),(j,l)\in\mathbb{N}^2} \hat{M}_{(i,k),(j,l)} r_{(i,k)}(s) r_{(i,k)}(t) r_{(j,l)}(s') r_{(j,l)}(t'),$$

and Assumption 7.4 is implied by the absolute summability of this matrix:

$$K = \sum_{(i,k),(j,l)\in\mathbb{N}^2} |\hat{M}_{(i,k),(j,l)}| = \sum_{i,j,k,l=1}^{\infty} |M_{ij} \cdot \lambda_k^{(i)} \hat{a}_{i,k}^2 \cdot \lambda_l^{(j)} \hat{a}_{j,l}^2| < \infty.$$

Next, we show two examples implying that the last two assumptions strictly generalize the d-component spin SK model.

Example 7.5. *Naturally, the d-component spin SK model*

$$\Gamma(s,t,s',t') = \langle\, s,t \,\rangle \langle\, s',t' \,\rangle = \sum_{i,j=1}^{d} s_i t_i s'_j t'_j.$$

fulfills both assumptions and hence is covered by this chapter for high enough temperature. Indeed, since in that case $\Sigma \subset \mathbb{R}^d$ is bounded, set $r := \sup_s \|s\|_1$ and let $r_i(s) := \frac{s_i}{r}$ for $i \leq d$ be given by the rescaled projection to the i-th component of s, and $a_i \equiv 0$ for $i > d$. By letting $M_{ij} := r^4$ compensate for this scaling as long as $i,j \leq d$, and $M_{ij} = 0$ when $i > d$ or $j > d$. Then, we are in the setting of the assumptions, at least after stipulating $\alpha_i(s,t) = a_i(s)a_i(t)$. And since all but a finite subset of the entries of $\mathbf{M}$ *are zero, this matrix certainly is summable, as well as the eigenvalues of the rank* 1 *functions α_i. Observe that in this case,* $\mathbf{M}$ *is of rank* 1*, but Γ_f again can have rank up to d.*

Another example is the following.

Example 7.6. *Let $\Sigma := [0,1]$ be equipped with p being the uniform measure,* $\mathbf{M}$ *being the diagonal matrix with diagonal $\frac{1}{\log(i)}$ and $a_i \in L^\infty(\Sigma,[0,1])$, e.g. $a_i(s) = 2^{-i}\, 1_{s>1/i}$, $i \in \mathbb{N}_{\geq 1}$. Then, $\alpha_i(s,t) := a_i(s)a_i(t)$ defines a GRF of infinite rank that fulfills Assumptions 7.3. If we let the $M_{ii} = \frac{1}{i^2}$ then Assumption 7.4 obviously holds as well.*

Last in this section, we introduce a lemma we will need several times, which generalizes the ubiquitous partial integration to this setting. See [11, Lemma 4.1] for a short proof.

Lemma 7.7 (∞-dimensional partial integration)**.** *Consider a probability space (T,p) and a centered Gaussian random field $\big(g(t)\big)_{t\in T}$ with covariance function Γ (and s.t. (T,d_Γ) is separable). Then, let z be a joint centered Gaussian variable with $\mathbb{E}z^2=1$ and denote the joint covariance by the function $C(t):=\operatorname{Cov}\big(z,g(t)\big)$ for all $t\in T$. Then:*

$$\mathbb{E}zF\bigg(\big(g(t)\big)_t\bigg)=\mathbb{E}\frac{\partial F}{\partial g}[C(\cdot)]=\mathbb{E}\frac{\partial}{\partial h}\bigg|_{h=0}F[g(\cdot)+h\,C(\cdot)],$$

that is the expectation of the Gâteaux directional derivative of F in direction $C(\cdot)$, if F is such that indeed its Gâteaux directional derivative and the expectation exist.

7.2. Order Parameters

As in the finite case, we need some $\pi(\cdot)$ and $\kappa(\cdot,\cdot)$. We will consider them to be densities w.r.t. p and $p^{\otimes 2}$, respectively. The fixed point equations are defined for $\kappa\in\mathcal{M}_+$:

$$\begin{aligned}\pi(s)=A_\kappa(s)&:=\mathbb{E}\tfrac{1}{Z_\kappa}e^{\beta g^{(\kappa)}(s)+\beta^2\Phi_\kappa(s)},\qquad Z_\kappa:=\operatorname{Tr}_s e^{\beta g^{(\kappa)}(s)+\beta^2\Phi_\kappa(s)}\\ \kappa(s,s')=B_\kappa(s,s')&:=\mathbb{E}\tfrac{1}{Z_\kappa^2}e^{\beta\left(g^{(\kappa)}(s)+g^{(\kappa)}(s')\right)+\beta^2\left(\Phi_\kappa(s)+\Phi_\kappa(s')\right)},\end{aligned}\tag{7.4}$$

where $g^{(\kappa)}$ is a GRF with covariance function $\mathbb{E}g^{(\kappa)}(s)g^{(\kappa)}(s')=\Gamma_\kappa(s,s')$ and

$$\pi(t)=\operatorname{Tr}_{t'}\kappa(t,t'),\quad \Phi_\kappa(s):=\tfrac12\operatorname{Tr}_t\gamma(s,t)\pi(t)-\tfrac12\operatorname{Tr}_{t,t'}\Gamma(s,t,s,t')\kappa(t,t').$$

By a simple argument we can see that $B_\bullet\colon\mathcal{M}(\Sigma^2)\to\mathcal{M}(\Sigma^2)$ is – if it exists – indeed a function with values in $\mathcal{M}_+$:

$$\sum_{i,j=1}^n a_ia_jB_\kappa(s_i,s_j)=\mathbb{E}\left(\sum_{i=1}^n a_i\frac{e^{\beta g^{(\kappa)}(s_i)+\beta^2\Phi_\kappa(s_i)}}{Z_\kappa}\right)^2\geq 0.$$

However, the existence of $B_\kappa(s,s')$ is a nasty problem, because we do not know the $g^{(\kappa)}$ to have bounded sample paths! Therefore, the Z_κ might not be bounded. Thus, we are bound to take the following:

Assumption 7.8. *Assume Γ and p are such that there is a solution $\kappa \in \mathcal{M}_+$ for the fixed point equation $\kappa = B_\kappa$ and that Γ_κ is the covariance function of a GRF with bounded paths.*

Unfortunately, we were not able to prove this in general, but only under Assumption 7.3. The issue lies buried in the fact that even for very simple Γ, Γ_f can be very general. Indeed, we have already seen in the d-component spin SK model on finite Σ that Γ_κ was of rank d even though Γ was of rank 1. Therefore, in the general case, it is not clear whether Γ_κ defines a GRF with bounded paths, and hence Z_κ is finite. In the case that Γ is such that Γ_f has bounded paths for some class of f, we will see in Corollary 7.12, that Assumption 7.8 holds indeed.

But first, we show the reason for taking Assumption 7.3.

Proposition 7.9. *Assume Assumption 7.3. Then, for each $f \in \mathcal{M}_+$ the GRF $g^{(f)}$ has p-essentially bounded paths.*

Proof. For the moment, let $i \in \mathbb{N}$ be fixed. Then, since $\gamma \in L^\infty$, hence also $\alpha_i \in L^\infty$. Thus, α_i defines as a Hilbert-Schmidt integral operator a compact, symmetric operator, and by the Spectral Theorem can be written as:

$$\alpha_i(s,t) = \sum_k \lambda_{i,k} v_{i,k}(s) v_{i,k}(t),$$

where $\lambda_{i,k} \geq 0$ and $v_{i,k} \in L^2$ is an orthonormal basis of eigenvectors. Then, again since $\alpha_i \in L^\infty$ and because of

$$v_{i,k}(s)^2 \leq \frac{1}{\lambda_{i,k}} \sum_{k'} \lambda_{i,k'} v_{i,k'}(s)^2 = \frac{1}{\lambda_{i,k}} \alpha_i(s,s),$$

it is clear that $v_{i,k} \in L^\infty$. Hence, $a_{i,k} := \sqrt{\lambda_{i,k}} v_{i,k} \in L_\infty$ and:

$$\alpha_i(s,t) = \sum_k a_{i,k}(s) a_{i,k}(t)$$

This calculation is now understood of as having been done for all $i \in \mathbb{N}$. Then, we have:

$$\begin{aligned}\Gamma_f(s,s') &= \mathrm{Tr}_{t,t'}\, f(t,t') \sum_{i,j} M_{ij}\alpha_i(s,t)\alpha_j(s,'t') \\ &= \sum_{i,j,k,l} a_{i,k}(s)a_{j,l}(s')M_{ij}\,\mathrm{Tr}_{t,t'}\, a_{i,k}(t)f(t,t')a_{j,l}(t') \\ &= \sum_{i,j,k,l} a_{i,k}(s)a_{j,l}(s')M'_{(ik),(jl)},\end{aligned}$$

where $M'_{(ik),(jl)} := M_{ij} \cdot \mathrm{Tr}_{t,t'}\, a_{i,k}(t)f(t,t')a_{j,l}(t')$. Now, $\mathbf{M}'$ is still positive definite since for all $n \in \mathbb{N}$ and all $(x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n}$:

$$\begin{aligned}\sum_{i,k,j,l=1}^{n} x_{ik}M'_{(ik),(jl)}x_{jl} &= \mathrm{Tr}_{t,t'}\, f(t,t') \sum_{i,j,k,l=1}^{n} a_{i,k}(t)x_{ik}M_{ij}a_{j,l}(t')x_{jl} \\ &= \mathrm{Tr}_{t,t'} \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} a_{i,k}(t)x_{ik}\right) f(t,t')M_{ij}\left(\sum_{l=1}^{n} a_{j,l}(t')x_{jl}\right) \\ &= \mathrm{Tr}_{t,t'} \sum_{i,j=1}^{n} x_i(t)f(t,t')M_{ij}x_j(t'),\end{aligned}$$

where $x_i(t) := \sum_{k=1}^n a_{i,k}(t)x_{ik}$. Now, by similar calculations as above, the symmetric positive definiteness of f gives:

$$f(t,t') = \sum_{k=1}^{\infty} \bar{f}_k(t)\bar{f}_k(t'),$$

with $\bar{f} \in L^\infty$. Then, this yields:

$$\sum_{i,j,k,l=1}^{n} x_{ik}M'_{(ik),(jl)}x_{jl} = \sum_{k=1}^{\infty}\sum_{i,j=1}^{n}\big(\mathrm{Tr}_t\, x_i(t)\bar{f}_k(t)\big)M_{ij}\big(\mathrm{Tr}_t\, x_j(t)\bar{f}_k(t)\big) \geq 0,$$

since M_{ij} is positive definite. Hence, there is a sequence of centered Gaussians

$(z_{ik})_{i,k=1}^{\infty}$ with covariance matrix $\mathbf{M}'$. Now, for all $i \in \mathbb{N}$:

$$\begin{aligned}\sum_{k=1}^{\infty} M'_{(ik),(ik)} &= \sum_{k=1}^{\infty} \mathrm{Tr}_{t,t'}\, a_{i,k}(t) f(t,t') a_{i,k}(t') M_{ii} \\ &= M_{ii}\ \mathrm{Tr}_{t,t'}\, f(t,t') \sum_{k=1}^{\infty} a_{i,k}(t) a_{i,k}(t') \\ &= M_{ii}\ \mathrm{Tr}_{t,t'}\, f(t,t')\alpha_i(t,t') \le M_{ii} \le \frac{L}{\log(i)}\end{aligned}$$

By reordering the $(z_{ik})_k$ (even with a different permutation for each $i \in \mathbb{N}$), we can assume that $(M'_{(ik),(ik)})_k$ is non-increasing for all $i \in \mathbb{N}$. Observe that for any non-increasing non-negative sequence $(b_k) \in \ell^1$ we have $kb_k \le \sum_{n=1}^{k} b_k \le \sum_{n=1}^{\infty} b_k < \infty$. Therefore, we have $M'_{(ik),(ik)} \le \frac{L}{k} \cdot \frac{1}{\log(i)}$. Now, we take the standard enumeration $(i_n, k_n)_n$ of the quadrant $\mathbb{N}^2$ for which holds $\sqrt{n} \le i_n + k_n$. This gives us:

$$M'_{(i_n k_n),(i_n k_n)} \le \frac{L}{k_n \cdot \log(i_n)} \le \frac{L}{\log(i_n + k_n)} \le \frac{2L}{\log n}$$

since $i + k \le i^k$ for all $i, k \ge 2$ and therefore $\sup_{i,k} z_{ik}$ is finite a.s. If we hence define $g^{(f)}(s) := \sum_{i,k=1}^{\infty} a_{i,k}(s) z_{i,k}$, $s \in \Sigma$, this has covariance matrix Γ_f and is a.s. bounded. □

Still, it seems awkward to think that Assumption 7.3 is of more than a technical reason and the boundedness of the sample paths could not be provable more generally than this. Indeed, since Γ_f is a convex combination of Γ, Slepian's Lemma might lead the way to show that the boundedness of Γ-fields implies the boundedness of the 'simpler' Γ_f-fields. Therefore, we give the maximal possible case as the following:

Conjecture 7.10. *If Γ is such that $g(\cdot,\cdot)$ has bounded sample paths $p^{\otimes 2}$-a.s. then Γ_f assures bounded sample paths for all $f \in \mathcal{M}_+$.*

This conjecture implies Assumption 7.8 by Corollary 7.12 at least for small enough β.

Now, we proceed to the argument why Conjecture 7.10 or Proposition 7.9 indeed imply Assumption 7.8, that is, that there exists a fixed point κ for β

small enough if Γ_f defines a bounded GRF for all $f \in \mathcal{M}_+$. We use the same techniques as in section 5.4. That is, we will prove that $B_\bullet$ is a contraction if β is small enough.

Proposition 7.11. *Assume Conjecture 7.10 or Assumption 7.3. Then, $B_\bullet$ is Lipschitz continuous with Lipschitz constant less or equal than $14\beta^2 K_\Gamma$.*

Proof. As in section 5.4, we will use Guerra interpolation once more. Let $\kappa, \kappa' \in \mathcal{M}_+$, and let $g := g^{(\kappa)}$ and $g' := g^{(\kappa')}$ be independent GRF's with covariance functions $\Gamma := \Gamma_\kappa$ and $\Gamma' := \Gamma_{\kappa'}$ resp. To lighten the notation, we will write $\gamma(t) := \Gamma(t,t)$, $\gamma'(t) := \Gamma'(t,t)$ and $\Phi := \Phi_\kappa$, $\Phi' := \Phi_{\kappa'}$.
Now, define the 'smart path':

$$\Pi_x(s) := \frac{1}{Z_x} \exp\left[\beta\sqrt{x} g(s) + \beta\sqrt{1-x} g'(s) + x\beta^2\Phi(s) + (1-x)\beta^2\Phi'(s)\right],$$
$$Z_x := \operatorname{Tr}_s e^{\beta\sqrt{x}g(s)+\beta\sqrt{1-x}g'(s)+\beta^2 x\Phi(s)+\beta^2(1-x)\Phi'(s)}$$

Then, a.s.:

$$\begin{aligned}\Pi'_x(s) := \beta\Pi_x(s)\cdot\Big[&\tfrac{1}{2\sqrt{x}}g(s) - \tfrac{1}{2\sqrt{1-x}}g'(s) + \Phi(s) - \Phi'(s)\\ &- \operatorname{Tr}_t \Pi_x(t)\Big(\tfrac{1}{2\sqrt{x}}g(t) - \tfrac{1}{2\sqrt{1-x}}g'(t) + \Phi(t) - \Phi'(t)\Big)\Big]\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}\frac{d}{dx}\mathbb{E}\Pi_x(s)\Pi_x(s') := \beta\mathbb{E}\,\Pi_x(s)\cdot\Pi_x(s')\cdot\Big[&\\ &\tfrac{1}{2\sqrt{x}}\big(g(s)+g(s')\big) - \tfrac{1}{2\sqrt{1-x}}\big(g'(s)+g'(s')\big)\\ &+\Phi(s)+\Phi(s')-\Phi'(s)-\Phi'(s')\\ &-2\operatorname{Tr}_t \Pi_x(t)\Big(\tfrac{1}{2\sqrt{x}}g(t) - \tfrac{1}{2\sqrt{1-x}}g'(t) + \Phi(t) - \Phi'(t)\Big)\Big]\end{aligned}$$

Now, we use the previous lemma for say $z = g(\hat{s})$ for a fixed $\hat{s} \in \Sigma$. Then, $C(s) = \Gamma(\hat{s}, s)$, and we let $F = \Pi_x(s)\Pi_x(s')$ be considered as a function of the GRFs $g(\cdot)$ and $g'(\cdot)$. Then, the partial integration amounts to calculate the

Gâteaux derivative of F in direction $C(\cdot)$:

$$\begin{aligned}
&\frac{d}{dh}\bigg|_{h=0} F\big(g(\cdot)+h\cdot C(\cdot)\big) \\
&= \frac{d}{dh}\bigg|_{h=0} \frac{1}{\left(\operatorname{Tr}_t e^{\beta\sqrt{x}\left(g(t)+hC(t)\right)+\beta\sqrt{1-x}g'(t)+\beta^2 x\Phi(t)+\beta^2(1-x)\Phi'(t)}\right)^2} \\
&\quad \cdot e^{\beta\sqrt{x}\left(g(s)+hC(s)+g(s')+hC(s')\right)+\beta\sqrt{1-x}\left(g'(s)+g'(s')\right)} \\
&\quad \cdot e^{\beta^2 x\left(\Phi(s)+\Phi(s')\right)+\beta^2(1-x)\left(\Phi'(s)+\Phi'(s')\right)} \\
&= \beta\sqrt{x}\Pi_x(s)\Pi_x(s')\cdot\Big[C(s)+C(s')-2\operatorname{Tr}_t \Pi_x(t)C(t)\Big] \\
&= \beta\sqrt{x}\Pi_x(s)\Pi_x(s')\operatorname{Tr}_t \Pi_x(t)\cdot\Big[\Gamma(\hat{s},s)+\Gamma(\hat{s},s')-2\Gamma(\hat{s},t)\Big]
\end{aligned}$$

Hence, if we implement this into the previous calculation, we obtain:

$$\begin{aligned}
\frac{d}{dx}\mathbb{E}\Pi_x(s)\Pi_x(s') = \tfrac{\beta^2}{2}\operatorname{Tr}_{t,t'}\mathbb{E}\,\Pi_x(s)\cdot\Pi_x(s')\cdot\Pi_x(t)\cdot\Pi_x(t')\cdot\Bigg[& \\
&\gamma(s)+\gamma(s')+2\Gamma(s,s')-2\Gamma(s,t')-2\Gamma(s',t') \\
&-\gamma'(s)-\gamma'(s')-2\Gamma'(s,s')+2\Gamma'(s,t')+2\Gamma'(s',t') \\
&+2\Phi(s)+2\Phi(s')-2\Phi'(s)-2\Phi'(s')-4\Phi(t)+4\Phi'(t) \\
&-2\Bigg(\gamma(t)+\Gamma(t,s)+\Gamma(t,s')-3\Gamma(t,t') \\
&-\gamma'(t)-\Gamma'(t,s)-\Gamma'(t,s')+3\Gamma'(t,t')\Bigg)\Bigg]
\end{aligned}$$

Now, since in this last expression all appearances of Γ are balanced, and by

$$\begin{aligned}
|\Gamma(s,s')-\Gamma'(s,s')| &\leq \operatorname{Tr}_{t,t'}|\Gamma(s,t,s',t')|\cdot|\kappa(t,t')-\kappa'(t,t')| \\
&\leq K_\Gamma\|\kappa-\kappa'\|_\infty,
\end{aligned}$$

and

$$\begin{aligned}
|\Phi(s)-\Phi'(s)| &\leq \operatorname{Tr}_{t,s'}\gamma(s,t)\cdot|\kappa(s,s')-\kappa'(s,s')| \\
&\quad + \operatorname{Tr}_{t,t'}|\Gamma(s,t,s,t')|\cdot|\kappa(t,t')-\kappa'(t,t')| \\
&\leq 2K_\Gamma\|\kappa-\kappa'\|_\infty,
\end{aligned}$$

we conclude that the Lipschitz constant of $B_\bullet$ in the ∞-norm is bounded by:

$$L \leq 14 \cdot \beta^2 \cdot K_\Gamma. \qquad \square$$

Corollary 7.12. *If Conjecture 7.10 or Assumption 7.3 are true and β is small enough, there is a unique fixed point κ. Hence, in that case, the fixed point equation has a unique solution and Assumption 7.8 holds. In particular, Assumption 7.3 implies Assumption 7.8.*

Proof. Since in that case by the previous proposition $B_\bullet$ is a contraction for β small enough, it remains to show that $\mathcal{M}_+$ is a closed subspace of $L^\infty(\Sigma^2)$. Hence, consider a sequence $f_1, f_2, \ldots \in \mathcal{M}_+$ converging to a $f \in L^\infty(\Sigma^2)$. We have to show $f \in \mathcal{M}_+$, that is, f is positive definite.
Let $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ and $s_1, \ldots, s_n \in \Sigma$. Then

$$\sum_{i,j=1}^{n} a_i a_j f(s_i, s_j) = \lim_{k\to\infty} \sum_{i,j=1}^{n} a_i a_j f_k(s_i, s_j).$$

Since all the f_k are positive definite, this limit is bounded by 0 from below. Hence, Γ_f is a positive definite function. $\square$

7.3. Example Calculation: Guerra's Interpolation

Henceforth assume Assumption 7.8. As a foreshadow to the generalization of the main proofs we now perform the Guerra interpolation in this setting. We start by defining the following smart path:

$$\begin{aligned} H_x(\boldsymbol{\sigma}) &:= \sqrt{x} H_N(\boldsymbol{\sigma}) + \sqrt{1-x}\beta \sum_i g_i(\sigma_i) + (1-x)\beta^2 \sum_i \Phi_\kappa(\sigma_i) \\ \varphi(x) &:= \tfrac{1}{N}\mathbb{E} \log \operatorname{Tr}_{\boldsymbol{\sigma}} e^{H_x(\boldsymbol{\sigma})} \end{aligned}$$

Then, we differentiate:

$$\begin{aligned} \varphi'(x) &= \tfrac{1}{N}\mathbb{E}\frac{1}{\operatorname{Tr}_{\boldsymbol{\sigma}} e^{H_x(\boldsymbol{\sigma})}} \frac{d}{dx} \operatorname{Tr}_{\boldsymbol{\sigma}} e^{H_x(\boldsymbol{\sigma})} = \tfrac{1}{N}\mathbb{E}\frac{1}{\operatorname{Tr}_{\boldsymbol{\sigma}} e^{H_x(\boldsymbol{\sigma})}} \operatorname{Tr}_{\boldsymbol{\sigma}} e^{H_x(\boldsymbol{\sigma})} \cdot H_x'(\boldsymbol{\sigma}) \\ &= \tfrac{1}{N}\nu_x \left[\tfrac{\beta}{2\sqrt{xN}} \sum_{i<j} g_{ij}(\sigma_i, \sigma_j) - \tfrac{\beta}{2\sqrt{1-x}} \sum_i g_i(\sigma_i) - \beta^2 \sum_i \Phi(\sigma_i) \right] \end{aligned}$$

Here, by using Fubini we exchanged some integrals, which is allowed since we have joint measurability as discussed in Remark 7.2. Now, as usual, we apply integration by parts. Here, this has to be done using Lemma 7.7 again by calculating the Gâteaux derivative. We obtain:

$$\begin{aligned}
\varphi'(x) &= \tfrac{\beta^2}{2N}\nu_x\Bigg[\tfrac{1}{N}\sum_{i<j}\big(\gamma(\sigma_i,\sigma_j)-\Gamma(\sigma_i,\sigma_j,\sigma_i',\sigma_j')\big) \\
&\qquad -\sum_i\big(\gamma_\kappa(\sigma_i)-\Gamma_\kappa(\sigma_i,\sigma_i')\big)\Bigg]-\tfrac{\beta^2}{N}\sum_i\nu_x\big[\Phi(\sigma_i)\big] \\
&= \tfrac{\beta^2}{2}\nu_x\Big[\int_{\Sigma^2}\gamma(s,t)\cdot\big(L_N(ds)-\pi(s)p(ds)\big)\big(L_N(dt)-\pi(t)p(dt)\big)\Big] \\
&\quad -\tfrac{\beta^2}{2}\nu_x\Big[\int_{\Sigma^4}\Gamma(s,t,s',t')\cdot\big(L_N(ds,ds')-\kappa(s,s')p(ds)p(ds')\big) \\
&\qquad \big(L_N(dt,dt')-\kappa(t,t')p(dt)p(dt')\big)\Big]-\tfrac{\beta^2}{2}\int_{\Sigma^2}\gamma(s,t)\pi(s)\pi(t)\,ds\,dt \\
&\quad +\tfrac{\beta^2}{2}\int_{\Sigma^4}\Gamma(s,t,s',t')\kappa(s,s')\pi(t,t')\,ds\,dt\,ds'\,dt' \\
&= \tfrac{\beta^2}{2}\int_{\Sigma^2}\gamma(s,t)\cdot\nu_x\Big[\big(L_N(ds)-\pi(s)p(ds)\big)\big(L_N(dt)-\pi(t)p(dt)\big)\Big] \\
&\quad -\tfrac{\beta^2}{2}\int_{\Sigma^4}\Gamma(s,t,s',t')\cdot\nu_x\Big[\big(L_N(ds,ds')-\kappa(s,s')p(ds)p(ds')\big) \\
&\qquad \big(L_N(dt,dt')-\kappa(t,t')p(dt)p(dt')\big)\Big]-\tfrac{\beta^2}{2}\int_{\Sigma^2}\gamma(s,t)\pi(s)\pi(t)\,ds\,dt \\
&\quad +\tfrac{\beta^2}{2}\int_{\Sigma^4}\Gamma(s,t,s',t')\kappa(s,s')\pi(t,t')\,ds\,dt\,ds'\,dt',
\end{aligned}$$

where $L_N(ds) := \frac{1}{N}\sum_i \delta_{\sigma_i}(ds)$ and $L_N(ds,ds') := \frac{1}{N}\sum_i \delta_{\sigma_i}(ds)\delta_{\sigma_i'}(ds')$ are the empirical measures in this setting. Here, we used $f(\sigma_i) = \int_\Sigma f(s)\delta_{\sigma_i}(ds)$ to obtain centered factors, which now turn out to be random signed measures. This complicates the matters in this setting and is the main difficulty of the generalization of the previous proofs.

Again, this formula for $\varphi'(x)$ is not easy to handle, because the order parameters κ actually depend on the point x of the interpolation. Therefore, we need again to apply the idea of [15] to prove our main theorems.

7.4. Generalizations of the Main Results

We now generalize the proofs of Chapter 5 to the GRF setting. We are bound to assume Assumption 7.8 and 7.4.

7.4.1. Preparation for Proofs

As previously, we will use a class of Hamiltonians $\mathcal{H}$ as we did in Section 4.1. Therefore, fix a dilution matrix $\mathbf{C}$ and the corresponding order parameters $\boldsymbol{\kappa} = \kappa_1, \ldots, \kappa_N \in \mathcal{M}_+$ that we assume to exist. Actually, Assumption 7.8 is formally not sufficient for this. Therefore, we assume in addition that all those fixed point equations have solutions, which of course we proved to be true for β small enough, if Conjecture 7.10 holds or under Assumption 7.3.

Of course all Hamiltonians $H_{\mathbf{A},\mathbf{B}}$ now are defined on much bigger spaces, but $\mathcal{H}_N(\mathbf{C}, \Gamma, p)$ is still parametrized just by $\mathbf{A}$, which here is the same as before! Hence, equations (4.5) and (4.6) are the same, simply reading the same notation now in our more general setting. This means that of course $\boldsymbol{\sigma}$, $g(\cdot,\cdot)$, κ, $g_i^{(j)}(\cdot)$, and $\Phi^{(j)}(\cdot)$ are all understood in the new setting.

7.4.2. Random Signed Measures

There is a big issue due to our heavy use of the Kronecker $\delta_i(s)$ in the proofs of Chapter 5. In the current setting, this has to be handled in the same way as in the end of the previous section on Guerra's interpolation. That is, the Kronecker δ's have to be replaced by Dirac measures. This of course changes the category of objects we are looking at drastically, even though the notation does not alter too much. The Kronecker δ's have to be handled more carefully from case to case.

One of our most used tools – analogously as in the finite case – will be to write:

$$f(\sigma_i) = \int_\Sigma f(s)\delta_{\sigma_i}(ds).$$

In such expressions, we will have to exchange integrals as in:

$$\nu_H\big(f(\sigma_i)\big) = \nu_H\Big(\int_\Sigma f(s)\delta_{\sigma_i}(ds)\Big) = \int_\Sigma f(s)\nu_H\big(\delta_{\sigma_i}(ds)\big).$$

This bears no problem, because the expectation of a random point measure $\nu_H\big(\delta_{\sigma_i}(ds)\big)$ is the law of the random variable σ_i under ν_H.
Still, this is not the end of the story. The point of introducing the indicators was to be able to factor out centered expressions. Inevitable, those then were random variables taking both positive and negative values. Therefore, when we now switch to random measures, they also will have to be signed ones:

$$\delta_{\sigma_i}(ds) - \pi_i(s)p(ds).$$

Observe that this is the difference of two probability measures, a random one and a deterministic one. To ease the notation, we hence introduce a shorthand for the following random signed measures:

$$\begin{aligned}\tilde{\delta}_i^{\ell}(ds) &:= \delta_{\sigma_i^{\ell}}(ds) - \pi_i(s)p(ds),\\ \tilde{\delta}_i^{\ell,\ell'}(ds, ds') &:= \delta_{(\sigma_i^{\ell}, \sigma_i^{\ell'})}(ds, ds') - \kappa_i(s, s')p(ds, ds')\end{aligned}$$

We will write often $\mu(A) := \int_A \mu(ds)$ for any signed measure μ and any measurable set A. Since $\tilde{\delta}_i$ is a random measure, the integral $\tilde{\delta}_i(A)$ is a random variable for all i. We will use often the following uniform bound of the variation of those signed random measures for all measurable A and all $\sigma_i \in \Sigma$:

$$\Big|\, |\tilde{\delta}_i|(A) \,\Big| = \left|\delta_i(A) + \int_A \pi(s)p(ds) - 2\pi(\sigma_i)p(\{\sigma_i\})\right| \leq 2,$$

which obviously holds in general for differences of probability measures. If p has no atoms then there is equality a.s. From this, one also sees easily

$$\left|\int f(s)\tilde{\delta}_i(ds)\right| \leq \int |f(s)|\, |\tilde{\delta}_i|(ds) \leq 2 \sup_{s\in\Sigma} |f(s)|.$$

The same obviously holds also for $\tilde{\delta}_i^{l,l'}$.

7.4.3. Generalization of Theorems 4.3 and 4.4

Now, we restate the main theorems we will prove in this setting. The formula for the free energy is the same – even though its meaning is viewed in the general setting – but we have a slower convergence rate:

Theorem 7.13 (GRF version of Theorem 4.3)**.** *Assume Assumption 7.4. The replica symmetric formula p_N given by* (4.10) *holds also in this setting, that is for all $H \in \mathcal{H}$:*

$$\left| \mathbb{E} \log \sum_{\boldsymbol{\sigma}} \exp\big(H(\boldsymbol{\sigma})\big) - p_N \right| \leq L \cdot \sum_i w_i.$$

The convergence of the overlap has to be restated in the measure theoretic setting:

Theorem 7.14 (GRF version of Theorem 4.4)**.** *Assume Assumption 7.4. For any given numbers $\alpha_1, \ldots, \alpha_N \geq 0$ and any $A_1, \ldots, A_N \in \mathcal{F}$ and $B_1, \ldots, B_N \in \mathcal{F}^{\otimes 2}$, we have:*

$$\nu_H\left[\left(\sum_{i \leq N} \alpha_i \tilde{\delta}_i(A_i)\right)^2\right] \leq 2L \sum_i \alpha_i^2 + 2L \left(\sum_i \alpha_i w_i\right)^2 \quad \textit{and}$$

$$\nu_H\left[\left(\sum_{i \leq N} \alpha_i \tilde{\delta}_i(B_i)\right)^2\right] \leq 2 \sum_i \alpha_i^2 + 2 \left(\sum_i \alpha_i w_i\right)^2$$

uniformly over all $H \in \mathcal{H}$.

7.5. Main Lemma

Fix Γ, p, $\mathbf{C}$, and $\boldsymbol{\kappa}$, where we assume both Assumptions 7.4 and 7.8. Correspondingly to the finite case, both Theorems 7.13 and 7.14 are consequences of the new Main Lemma 7.15. To formulate this we will use the following redefined quantities where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N_{\geq 0}$:

$$U(\boldsymbol{\alpha}) := \sup_{\substack{a_i \colon \Sigma \to [0,1] \\ \mathcal{F}\text{-m.b.}}} \sup_{H \in \mathcal{H}} \sqrt{\nu_H\left[\left(\sum_i \alpha_i \int_\Sigma a_i(\eta_i) \tilde{\delta}_i(d\eta_i)\right)^2\right]},$$

$$V(\boldsymbol{\alpha}) := \sup_{\substack{b_i \colon \Sigma^2 \to [0,1] \\ \mathcal{F}^{\otimes 2}\text{-m.b.}}} \sup_{H \in \mathcal{H}} \sqrt{\nu_H\left[\left(\sum_i \alpha_i \int_{\Sigma^2} b_i(\eta_i, \eta_i') \tilde{\delta}_i^{1,2}(d\eta_i, d\eta_i')\right)^2\right]}.$$

Here the maximum over all configurations $\boldsymbol{\eta}, \boldsymbol{\eta}'$ has to be enlarged heavily to a supremum over all measurable functions $b_i \colon \Sigma^2 \to [0,1]$. The discrete case we had previously here can be recovered by use of indicator functions, c.f. the proof of Theorem 7.14.

For any fixed $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N_{\geq 0}$, we will also set for $\boldsymbol{\sigma}, \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \Sigma^N$ and for all measurable sequences of functions $a, a_1, \dots, a_N \colon \Sigma \to [0,1]$ and $b, b_1, \dots, b_N \colon \Sigma^2 \to [0,1]$:

$$F_i(a; a_1, \dots, a_N; \boldsymbol{\sigma}) := \int_\Sigma a(s)\tilde{\delta}_i(ds) \cdot \sum_{j=1}^N \alpha_j \int_\Sigma a_j(\eta_j)\tilde{\delta}_j(d\eta_j),$$

$$G_i(b; b_1, \dots, b_N; \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) := \int_{\Sigma^4} b(s, s')\tilde{\delta}_i^{1,2}(ds, ds') \cdot \sum_{j=1}^N \alpha_j b_j(\eta_j, \eta_j')\tilde{\delta}_j^{1,2}(d\eta_j, d\eta_j').$$

As in the finite case those quantities are linked to the definition of $U(\boldsymbol{\alpha})$ and $V(\boldsymbol{\alpha})$ via the simple calculation:

$$\sum_{i=1}^N \alpha_i F_i(a_i; a_1, \dots, a_N; \boldsymbol{\sigma}) = \sum_{i=1}^N \alpha_i \int a_i(s)\tilde{\delta}_i(ds) \cdot \sum_{j=1}^N \alpha_j \int a_j(t)\tilde{\delta}_j(dt)$$

With the new definitions, we now will get up to a different constant the same formulas for the upper bounds as in the finite setting Main Lemma 5.1.

Lemma 7.15 (GRF version of Main Lemma). *Assume Assumption 7.4 and let $H = H_{\mathbf{A},\mathbf{B}} \in \mathcal{H}$. Then, for all $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N_{\geq 0}$ and for all measurable functions $a, a_1, \dots, a_N \colon \Sigma \to [0,1]$ and $b, b_1, \dots, b_N \colon \Sigma^2 \to [0,1]$ we have:*

$$\left|\nu_H\Big[F_i(a; a_1, \dots, a_N; \boldsymbol{\sigma})\Big]\right| \leq L\alpha_i + 2L \cdot K \cdot U(\boldsymbol{\alpha}) \cdot \big(U(a_{\bullet i}) + V(a_{\bullet i})\big),$$

$$\left|\nu_H\Big[G_i(b; b_1, \dots, b_N; \boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\Big]\right| \leq L\alpha_i + 8L \cdot K \cdot V(\boldsymbol{\alpha}) \cdot \big(U(a_{\bullet i}) + V(a_{\bullet i})\big),$$

where we recall that $a_{\bullet i}$ is the i-th row of $\mathbf{A}$ given by $H = H_{\mathbf{A},\mathbf{B}}$.

7.5.1. Proof of Lemma 7.15

Precisely as before, this lemma is proved using the smart path $(H_x)_{x\in[0,1]} \subset \mathcal{H}$ that decouples the last spin from the others, by fading in the interaction with the i-th spin in the Hamiltonian H. Large parts here are just copies of the proof in Section 5.1. The most important changes after having redefined the symbols are in the calculation of $\varphi(0)$ and $\psi(0)$, then the differentiation by Gâteaux derivative – which leads to the same formulas – and last, but not least delicate, the Cauchy-Schwarz step. For sake of readability, we will keep the identical parts in this chapter, although we could just refer back to Section 5.1.1.

Fix measurable functions $a, a_1, \ldots, a_N \colon \Sigma \to [0,1]$ and $b, b_1, \ldots, b_N \colon \Sigma^2 \to [0,1]$ and the Hamiltonian $H = H_{\mathbf{A},\mathbf{B}} \in \mathcal{H}$. Further, we prove the lemma for $i = N$. Hence again:

$$F(\boldsymbol{\sigma}) := F_N(a; a_1, \ldots, a_N; \boldsymbol{\sigma}), \qquad G(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := G_N(b; b_1, \ldots, b_N; \boldsymbol{\sigma}, \boldsymbol{\sigma}'),$$
$$\Phi^{(i)}(s) := \Phi_{\kappa_i}(s), \qquad \Gamma^{(i)}(s,s') := \Gamma_{\kappa_i}(s,s'), \qquad \gamma^{(i)}(s) := \Gamma^{(i)}(s,s).$$

Identically to the finite case, we define $a_{ij}(x), b_{ij}(x)$, $x \in [0,1]$ by:

$$a_{ij}(x) := \begin{cases} a_{ij} & i < j < N \\ x a_{ij} & i < j = N \end{cases}, \quad b_{ij}(x) := b_{ij} + \begin{cases} (1-x) a_{ij} & i = N \text{ or } j = N \\ 0 & i, j < N \end{cases} \tag{7.5}$$

and let $H_x := H_{\mathbf{A}(x),\mathbf{B}(x)}$ and $\nu_x(\cdot) := \nu_{H_x}(\cdot)$. Let:

$$\varphi_N(x) := \nu_x[F(\boldsymbol{\sigma})] = \nu_x[F_N(a; a_1, \ldots, a_N; \boldsymbol{\sigma})],$$
$$\psi_N(x) := \nu_x[G(\boldsymbol{\sigma}, \boldsymbol{\sigma}')] = \nu_x[G_N(b; b_1, \ldots, b_N; \boldsymbol{\sigma}, \boldsymbol{\sigma}')].$$

The first question we have to check in the generalized setting is whether we can handle the starting points:

$$\begin{aligned} \varphi(0) = \alpha_N \nu_0 &\Big[\int_{\Sigma^2} a(s) a_N(\eta_N) \tilde{\delta}_N(ds) \cdot \tilde{\delta}_N(d\eta_N) \Big] \\ &+ \sum_{i<N} \alpha_i \nu_0 \Big[\int_{\Sigma^2} a(s) a_i(\eta_N) \tilde{\delta}_N(ds) \tilde{\delta}_i(d\eta_i) \Big] \end{aligned} \tag{7.6}$$

$$\begin{aligned} \psi(0) = \alpha_N \nu_0 &\Big[\int_{\Sigma^4} b(s,s') b_N(\eta_N, \eta'_N) \tilde{\delta}_N^{1,2}(ds, ds') \cdot \tilde{\delta}_N^{1,2}(d\eta_N, d\eta'_N) \Big] \\ &+ \sum_{i<N} \alpha_i \nu_0 \Big[\int_{\Sigma^4} b(s,s') b_N(\eta_N, \eta'_N) \tilde{\delta}_N^{1,2}(ds, ds') \tilde{\delta}_i^{1,2}(d\eta_i, d\eta'_i) \Big] \end{aligned} \tag{7.7}$$

In both cases the first summand can be bound by $4\alpha_N$ because both measures have variation at most 2. The second summands more interesting. According to (7.5), the last spin is decoupled completely from the others. For $\psi(0)$, clearly κ_N fulfills the fixed point equation (7.4), thus in this case the second summand in (7.7) becomes:

$$\begin{aligned}\nu_0\Big[\int_{\Sigma^4} & b(s,s')b_N(\eta_N,\eta'_N)\tilde{\delta}^{1,2}_N(ds,ds')\tilde{\delta}^{1,2}_i(d\eta_i,d\eta'_i)\Big] \\ &= \int_{\Sigma^4} b(s,s')b_N(\eta_N,\eta'_N)\nu_0\Big[\tilde{\delta}^{1,2}_N(ds,ds')\Big]\cdot\nu_0\Big[\tilde{\delta}^{1,2}_i(d\eta_i,d\eta'_i)\Big]\end{aligned}$$

vanishes, because the law of σ^1_N,σ^2_N under ν_0 is given by the density κ_N w.r.t. $p^{\otimes 2}$ and we can exchange integrations for random point measures. Here we used that b and b_i are non-random. For the second summand of $\varphi(0)$ in (7.6) the fixed point equation for κ_N again implies:

$$\tilde{\delta}_N(dt) = \delta_N(dt) - \pi_N(t)\,p(dt) = \int_\Sigma \big[\delta_N(dt)\delta_N(dt') - \kappa_N(t,t')\,p(dt)p(dt')\big],$$

which vanishes by the same fixed point expression under ν_0.

Differentiation and Integration by Parts We calculate the derivative of $\varphi(x)$ in the GRF setting by the Gâteaux derivative. This will give the same formula as in the finite case. First, the derivative certainly exists, since the only dependence on x in

$$\varphi_N(x) = \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\,F(\boldsymbol{\sigma})\frac{e^{H_x(\boldsymbol{\sigma})}}{Z},$$

is in the second factor of the integrand and everything is bounded. As in the finite case, we calculate:

$$\begin{aligned}\varphi'_N(x) &= \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}}\,F(\boldsymbol{\sigma})\frac{e^{H_x(\boldsymbol{\sigma})}\cdot H'_x(\boldsymbol{\sigma})\cdot Z - e^{H_x(\boldsymbol{\sigma})}\cdot Z'}{Z^2} \\ &= \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2}\,F(\boldsymbol{\sigma}^1)\cdot\big(H'_x(\boldsymbol{\sigma}^1) - H'_x(\boldsymbol{\sigma}^2)\big)\cdot\frac{e^{H_x(\boldsymbol{\sigma}^1)+H_x(\boldsymbol{\sigma}^2)}}{Z^2} \\ &= \nu_x\Big[F(\boldsymbol{\sigma}^1)\cdot\big(H'_x(\boldsymbol{\sigma}^1) - H'_x(\boldsymbol{\sigma}^2)\big)\Big],\end{aligned}$$

and we have as well:

$$H'_x(\boldsymbol{\sigma}) = \frac{1}{2\sqrt{x}} \sum_{i<N} \sqrt{a_{iN}} g_{iN}(\sigma_i, \sigma_N) - \sum_{i<N} a_{iN} \cdot \Big(\\ \tfrac{1}{2\sqrt{b_{iN}(x)}} g_i^{(N)}(\sigma_i) + \tfrac{1}{2\sqrt{b_{iN}(x)}} g_N^{(i)}(\sigma_N) + \Phi^{(N)}(\sigma_i) + \Phi^{(i)}(\sigma_N)\Big)$$

Now, we have to use the Gâteaux derivative version of integration by parts in Lemma 7.7

$$\nu_x[F(\boldsymbol{\sigma}) \cdot g_{iN}(\sigma_i, \sigma_N)] = \mathbb{E} \frac{\partial \left\langle \ F(\boldsymbol{\sigma}) \, \frac{\exp H_x(\boldsymbol{\sigma})}{Z} \ \right\rangle}{\partial g} [\Gamma(\sigma_i, \sigma_N, \cdot, \cdot)].$$

We now calculate this Gâteaux derivative, which again exists by interchange of limits:

$$\begin{aligned}
&\frac{\partial}{\partial h}\bigg|_{h=0} \operatorname{Tr}_{\boldsymbol{\sigma}} F(\boldsymbol{\sigma}) \frac{\exp\left[H_x(\boldsymbol{\sigma}) + \sqrt{a_{ij}(x)}\, h\, \gamma(\sigma_i, \sigma_j)\right]}{\operatorname{Tr}_{\sigma'} \exp\left[H_x(\boldsymbol{\sigma}') + \sqrt{a_{ij}(x)}\, h\, \Gamma(\sigma_i, \sigma_j, \sigma'_i, \sigma'_j)\right]} \\
&= \operatorname{Tr}_{\boldsymbol{\sigma}} F(\boldsymbol{\sigma}) \frac{\operatorname{Tr}_{\sigma'} e^{H_x(\boldsymbol{\sigma}) + H_x(\boldsymbol{\sigma}')} \cdot \sqrt{a_{ij}(x)}\left[\gamma(\sigma_i, \sigma_j) - \Gamma(\sigma_i, \sigma_j, \sigma'_i, \sigma'_j)\right]}{Z^2} \\
&= \left\langle \ F(\boldsymbol{\sigma}) \cdot \sqrt{a_{ij}(x)}\left[\gamma(\sigma_i, \sigma_j) - \Gamma(\sigma_i, \sigma_j, \sigma'_i, \sigma'_j)\right] \ \right\rangle_x
\end{aligned}$$

This obviously gives the same formula as in the finite case. Therefore we are happy to refer the reader to Section 5.1.1 for the details of the calculation and arrive at:

$$\begin{aligned}
\varphi'_N(x) &= \tfrac{1}{2}\nu_x\left[F(\boldsymbol{\sigma})\big(v(1) - v(2) - 2v(1,2) + 2v(2,3)\big)\right], \\
\psi'_N(x) &= \tfrac{1}{2}\nu_x\Bigg[G(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\bigg(v(1) + v(2) - 2v(3) \\
&\qquad + 2v(1,2) - 4v(1,3) - 4v(2,3) + 6v(3,4)\bigg)\Bigg]
\end{aligned}$$

where we use the identical notation for $l \neq l'$ as in the finite case:

$$v(l) := \sum_{i<N} a_{iN} \left[\gamma(\sigma_i^l, \sigma_N^l) - \gamma(\sigma_i^l, \pi_N) - \gamma(\sigma_N^l, \pi_i) \right],$$
$$v(l, l') := \sum_{i<N} a_{iN} \left[\Gamma(\sigma_i^l, \sigma_N^l, \sigma_i^{l'}, \sigma_N^{l'}) - \Gamma^{(N)}(\sigma_i^l, \sigma_i^{l'}) - \Gamma^{(i)}(\sigma_N^l, \sigma_N^{l'}) \right]$$

Reordering of terms Now, we use the Dirac measure trick on $v(l, l')$:

$$\begin{aligned} v(l, l') = \sum_{i<N} a_{iN} \int_{\Sigma^4} \Gamma(s, t, s,' t') \Big[& \big(\delta_{(\sigma_i^l, \sigma_i^{l'})}(ds, ds') - \kappa_i(s, s')\, p^{\otimes 2}(ds, ds')\big) \\ & \cdot \big(\delta_{(\sigma_N^l, \sigma_N^{l'})}(dt, dt') - \kappa_N(dt, dt')\, p^{\otimes 2}(dt, dt')\big) \\ & - \kappa_i(s, s')\kappa_N(t, t') p^{\otimes 4}(ds, ds', dt, dt') \Big] \\ = \sum_{i<N} a_{iN} \int_{\Sigma^4} \Gamma(s, t, s,' t') \cdot \Big[& \tilde{\delta}_i^{l,l'}(ds, ds') \cdot \tilde{\delta}_N^{l,l'}(dt, dt') \\ & - \kappa_i(s, s')\kappa_N(t, t') p^{\otimes 4}(ds, ds', dt, dt') \Big] \end{aligned}$$

Observe how the integration over the random signed measures emerged. Hence we define analogously to the finite case:

$$\bar{v}(l) := \sum_{i<N} a_{iN} \int_{\Sigma^2} \gamma(s, t) \left[\tilde{\delta}_i^l(ds) \cdot \tilde{\delta}_N^l(dt) \right]$$
$$\bar{v}(l, l') := \sum_{i<N} a_{iN} \int_{\Sigma^4} \Gamma(s, t, s,' t') \left[\tilde{\delta}_i^{l,l'}(ds, ds') \cdot \tilde{\delta}_N^{l,l'}(dt, dt') \right],$$

yielding:

$$\varphi'_N(x) = \tfrac{1}{2}\nu_x \left[F(\boldsymbol{\sigma}) \Big(\bar{v}(1) - \bar{v}(2) - 2\bar{v}(1, 2) + 2\bar{v}(2, 3) \Big) \right],$$
$$\begin{aligned} \psi'_N(x) = \tfrac{1}{2}\nu_x \Big[G(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \Big(& \bar{v}(1) + \bar{v}(2) - 2\bar{v}(3) \\ & + 2\bar{v}(1, 2) - 4\bar{v}(1, 3) - 4\bar{v}(2, 3) + 6\bar{v}(3, 4) \Big) \Big]. \end{aligned}$$

Again we use the following notation for $l \neq l'$:

$$F(l,l') := \tfrac{1}{2}\nu_x\big[F(\boldsymbol{\sigma}) \cdot \bar{v}(l,l')\big], \qquad F(l) := \tfrac{1}{2}\nu_x\big[F(\boldsymbol{\sigma}) \cdot \bar{v}(l)\big],$$
$$G(l,l') := \tfrac{1}{2}\nu_x\big[G(\boldsymbol{\sigma},\boldsymbol{\sigma}') \cdot \bar{v}(l,l')\big], \qquad G(l) := \tfrac{1}{2}\nu_x\big[G(\boldsymbol{\sigma},\boldsymbol{\sigma}') \cdot \bar{v}(l)\big],$$

which is for $l \neq l'$:

$$\begin{aligned} F(l,l') &= \tfrac{1}{2}\nu_x\bigg[\sum_{i\le N}\alpha_i\int_{\Sigma^2} a(t)a_i(\eta_i)\tilde{\delta}_N^1(dt)\tilde{\delta}_i^1(d\eta_i) \\ &\qquad \cdot \int_{\Sigma^4}\Gamma(s,t,s',t')\,\tilde{\delta}_N^{l,l'}(dt,dt')\sum_{i<N}a_{iN}\tilde{\delta}_i^{l,l'}(ds,ds')\bigg], \\ G(l,l') &= \tfrac{1}{2}\nu_x\bigg[\sum_{i\le N}\alpha_i\int_{\Sigma^4} b(t,t')b_i(\eta_i,\eta_i')\tilde{\delta}_N^{1,2}(dt,dt')\tilde{\delta}_i^{1,2}(d\eta_i,d\eta_i') \\ &\qquad \cdot \int_{\Sigma^4}\Gamma(s,t,s',t')\,\tilde{\delta}_N^{l,l'}(dt,dt')\sum_{i<N}a_{iN}\tilde{\delta}_i^{l,l'}(ds,ds')\bigg]. \end{aligned} \tag{7.8}$$

Cauchy-Schwarz Again, we will use the Cauchy-Schwarz inequality for this product. This is where the dependence on Assumption 7.4 is raised. Fix $\hat{l},\hat{l}'$. Observe that we now have:

$$\begin{aligned} G(\hat{l},\hat{l}') &= \tfrac{1}{2}\nu_x\big[X\cdot Y\big], \qquad \text{where} \\ X &:= \int_{\Sigma^2} b(t,t')\tilde{\delta}_N^{1,2}(dt,dt')\cdot\sum_{i\le N}\alpha_i\int_{\Sigma^2} b_i(\eta_i,\eta_i')\tilde{\delta}_i^{1,2}(d\eta_i,d\eta_i') \quad \text{and} \\ Y &:= \int_{\Sigma^4}\Gamma(s,t,s',t')\,\tilde{\delta}_N^{\hat{l},\hat{l}'}(dt,dt')\sum_{i<N}a_{iN}\tilde{\delta}_i^{\hat{l},\hat{l}'}(ds,ds'). \end{aligned}$$

Because we assume Assumption 7.4, we can rewrite this as:

$$\begin{aligned} \nu_x\big[X\cdot Y\big] = \sum_{k,l} M_{kl}\nu_x\Big[X\cdot\int_{\Sigma^4} r_k(t)r_l(t')\tilde{\delta}_N^{\hat{l},\hat{l}'}(dt,dt') \\ \cdot r_k(s)r_l(s')\sum_{i<N}a_{iN}\tilde{\delta}_i^{\hat{l},\hat{l}'}(ds,ds')\Big] \end{aligned}$$

Here we exchanged the infinite sum with the expectation and the integral because the integrands are bounded and the signed measures have variation bounded by 2. Hence, we can apply Cauchy-Schwarz inequality for X and the integral, and obtain

$$
\begin{aligned}
\Big|\nu_x\big[X\cdot Y\big]\Big| &\le \sum_{k,l}|M_{kl}|\,\sqrt{\nu_x[X^2]}\cdot\Bigg\{\nu_x\Bigg[\Big(\int_{\Sigma^2} r_k(t)r_l(t')\tilde{\delta}_N^{\hat{l},\hat{l}'}(dt,dt')\Big)^2 \\
&\qquad \cdot\Big(\int_{\Sigma^2} r_k(s)r_l(s')\sum_{i<N} a_{iN}\tilde{\delta}_i^{\hat{l},\hat{l}'}(ds,ds')\Big)^2\Bigg]\Bigg\}^{\frac{1}{2}} \\
&\le \sqrt{\nu_x[X^2]}\cdot\sum_{k,l}|M_{kl}|\sqrt{\nu_x\Big[\Big(\int_{\Sigma^2} r_k(s)r_l(s')\sum_{i<N} a_{iN}\tilde{\delta}_i^{\hat{l},\hat{l}'}(ds,ds')\Big)^2\Big]} \\
&\qquad \cdot\sup_{\boldsymbol{\sigma}^l,\boldsymbol{\sigma}^{l'}}\Big|\int_{\Sigma^2} r_k(t)r_l(t')\tilde{\delta}_N^{\hat{l},\hat{l}'}(dt,dt')\Big| \\
&\le 2K\cdot\sqrt{\nu_x[X^2]}\cdot V(\mathbf{a}_{\bullet N})
\end{aligned}
\tag{7.9}
$$

since $f(s,s') := r_k(s)r_k(s')$ is a $\mathcal{F}^{\otimes 2}$-m.b. function with values in $[0,1]$ and $H_x \in \mathcal{H}$. On the other hand:

$$
\begin{aligned}
\nu_x[X^2] \le \sup_{\boldsymbol{\sigma}^l,\boldsymbol{\sigma}^{l'}}&\Big(\int_{\Sigma^2} b(t,t')\tilde{\delta}_N^{\hat{l},\hat{l}'}(dt,dt')\Big)^2 \\
&\cdot\nu_x\Big[\Big(\int_{\Sigma^2}\sum_{i\le N}\alpha_i b_i(\eta_i,\eta_i')\tilde{\delta}_i^{\hat{l},\hat{l}'}(d\eta_i,d\eta_i')\Big)^2\Big] \le 4V(\boldsymbol{\alpha})^2.
\end{aligned}
$$

This gives us with analogous arguments:

$$
\begin{aligned}
\big|F(l)\big| &\le LU(\boldsymbol{\alpha})\cdot U(\mathbf{a}_{\bullet N}), &\qquad \big|F(l,l')\big| &\le LU(\boldsymbol{\alpha})\cdot V(\mathbf{a}_{\bullet N}),\\
\big|G(l)\big| &\le LV(\boldsymbol{\alpha})\cdot U(\mathbf{a}_{\bullet N}), &\qquad \big|G(l,l')\big| &\le LV(\boldsymbol{\alpha})\cdot V(\mathbf{a}_{\bullet N}).
\end{aligned}
$$

Now, we can wrap this up as in the finite case:

$$
\begin{aligned}
\big|\varphi_N'(x)\big| &\le 2L\,U(\boldsymbol{\alpha})\Big[U(\mathbf{a}_{\bullet N})+2V(\mathbf{a}_{\bullet N})\Big] \le L\,U(\boldsymbol{\alpha})\cdot\big(U(\mathbf{a}_{\bullet N})+V(\mathbf{a}_{\bullet N})\big),\\
\big|\psi_N'(x)\big| &\le 4L\,U(\boldsymbol{\alpha})\Big[U(\mathbf{a}_{\bullet N})+4V(\mathbf{a}_{\bullet N})\Big] \le L\,V(\boldsymbol{\alpha})\cdot\big(U(\mathbf{a}_{\bullet N})+V(\mathbf{a}_{\bullet N})\big),
\end{aligned}
$$

and obtain:

$$\big|\varphi_N(1)\big| \le L\alpha_N + 2\cdot L\cdot U(\alpha)\cdot\Big(U(\mathbf{a}_{\bullet N}) + V(\mathbf{a}_{\bullet N})\Big),$$
$$\big|\psi_N(1)\big| \le L\alpha_N + 8\cdot L\cdot V(\alpha)\cdot\Big(U(\mathbf{a}_{\bullet N}) + V(\mathbf{a}_{\bullet N})\Big).$$

Hence, this proves Lemma 7.15. □

7.6. Proofs of Theorems 7.13 and 7.14

Now that we have got Main Lemma 7.15, things are almost the same as in the finite case since we have again:

$$U(\boldsymbol{\alpha})^2 = \sup_{\substack{a_i:\Sigma\to[0,1]\\ \mathcal{F}\text{-m.b.}}}\ \sup_{H\in\mathcal{H}} \sum_i \alpha_i \nu_H\Big[\int_\Sigma a_i(s)\tilde{\delta}_i(ds)\cdot\sum_j \alpha_j \int_\Sigma a_j(\eta_j)\tilde{\delta}_j(d\eta_j)\Big],$$

$$V(\boldsymbol{\alpha})^2 = \sup_{b_i:\ \Sigma^2\to[0,1]\ \mathcal{F}^{\otimes 2}\text{-m.b.}}\ \sup_{H\in\mathcal{H}} \sum_i \alpha_i \nu_H\Big[\int_{\Sigma^2} b_i(s,s')\tilde{\delta}_i^{1,2}(ds,ds')$$
$$\cdot \sum_j \alpha_j \int_{\Sigma^2} b_j(\eta_j,\eta_j')\tilde{\delta}_j^{1,2}(d\eta_j,d\eta_j')\Big]$$

In fact, only very little has to be changed for the GRF setting. First, there has to be stated a sentence in the proof of Theorem 7.14. And second, we will use the same calculation as in the previous Cauchy-Schwarz step for the proof of Theorem 7.13, therefore this also again depends on Assumption 7.4. This allows us to abstain from generalizing Lemma 5.3 at the expense of the different convergence rate in the Theorem.

The start of this section up to Corollary 7.16 and its proof is the identical argument as in the finite case. That is, if we set

$$v_i := \tfrac{1}{2}\Big(\sup U(\boldsymbol{\beta}) + \sup V(\boldsymbol{\beta})\Big), \qquad \forall i \le N,$$

where the suprema are over all sequences $\boldsymbol{\beta}$ s.t. $0 \le \beta_j \le c_{ij}$, then Lemma

7.15 gives:

$$U(\boldsymbol{\alpha})^2 \leq \sum_i \alpha_i^2 + 4K \cdot U(\boldsymbol{\alpha}) \sum_i \alpha_i v_i$$
$$V(\boldsymbol{\alpha})^2 \leq \sum_i \alpha_i^2 + 16 \cdot K \cdot V(\boldsymbol{\alpha}) \sum_i \alpha_i v_i$$

and we get again

$$\max\{U(\boldsymbol{\alpha}), V(\boldsymbol{\alpha})\} \leq \sqrt{\sum_i \alpha_i^2} + 16 \cdot K \cdot \sum_i \alpha_i v_i, \tag{7.10}$$

which gives the same formula as in Corollary 5.2, where one has to keep in mind that the definitions of $U(\boldsymbol{\alpha})$ and $V(\boldsymbol{\alpha})$ differ!

Corollary 7.16. *Under the assumptions of Lemma 7.15, for all $i \leq N$:*

$$v_i \leq w_i,$$

where we recall the definition of $\mathbf{w} = (w_1, \ldots, w_N)$ in (4.8).

Proof. This is just the same as in the proof of Corollary 5.2 □

Proof of Theorem 7.14. Just apply the previous corollary on (7.10) using the indicator functions:

$$a_i := 1_{A_i} \qquad \text{and} \qquad b_i := 1_{B_i}.$$ □

Proof of Theorem 7.13. We again look at the same smart path as in the finite case:

$$H_x(\boldsymbol{\sigma}) := \sum_{i<j} \sqrt{x a_{ij}} g_{ij}(\sigma_i, \sigma_j) + \sum_{i,j} \sqrt{b_{ij} + (1-x) a_{ij}} g_i^{(j)}(\sigma_i)$$
$$+ \sum_{i,j} \big(b_{ij} + (1-x) a_{ij}\big) \Phi^{(j)}(\sigma_i).$$

Then, $H_x \in \mathcal{H}$ for all $x \in [0,1]$. Let $\varphi(x) := \mathbb{E}\log(\mathrm{Tr}_{\boldsymbol{\sigma}}\, e^{H_x(\boldsymbol{\sigma})})$. By the same calculations as before we get the derivative w.r.t. x:

$$\begin{aligned} H'_x(\boldsymbol{\sigma}) = \sum_{i<j} \frac{\sqrt{a_{ij}}}{2\sqrt{x}} g_{ij}(\sigma_i,\sigma_j) - \sum_{i,j} \frac{a_{ij}}{2\sqrt{b_{ij}+(1-x)a_{ij}}} g_i^{(j)}(\sigma_i) \\ - \sum_{i,j} a_{ij}\Phi^{(j)}(\sigma_i). \end{aligned}$$

This yields:

$$\begin{aligned} \varphi'(x) &= \mathbb{E}\,\mathrm{Tr}_{\boldsymbol{\sigma}} \frac{e^{H_x(\boldsymbol{\sigma})}\cdot H'_x(\boldsymbol{\sigma})}{Z} = \nu_x\Big[H'_x(\boldsymbol{\sigma})\Big] \\ &= \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \Gamma(\sigma_i,\sigma_j,\sigma'_i,\sigma'_j) - \gamma^{(j)}(\sigma_i) + \Gamma^{(j)}(\sigma_i,\sigma'_i) \\ &\qquad - \gamma^{(i)}(\sigma_j) + \Gamma^{(i)}(\sigma_j,\sigma'_j) - 2\Phi^{(i)}(\sigma_j) - 2\Phi^{(j)}(\sigma_i)\Big] \end{aligned}$$

and by the definitions of $\Gamma^{(j)}(s,s') = \Gamma_{\kappa_j}(s,s')$:

$$\begin{aligned} \varphi'(x) &= \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \Gamma(\sigma_i,\sigma_j,\sigma'_i,\sigma'_j) + \Gamma^\star(\sigma_i,\sigma'_i;\kappa_j) \\ &\qquad + \Gamma^\star(\sigma_j,\sigma'_j;\kappa_i) - \gamma(\sigma_j,\pi_i) - \gamma(\sigma_i,\pi_j)\Big]. \\ &= \frac{1}{4}\sum_{i,j} a_{ij}\nu_x\Big[\gamma(\sigma_i,\sigma_j) - \gamma(\sigma_j,\pi_i) - \gamma(\sigma_i,\pi_j) + \gamma(\pi_i,\pi_j) \\ &\qquad - \Gamma(\sigma_i,\sigma_j,\sigma'_i,\sigma'_j) + \Gamma^\star(\sigma_i,\sigma'_i;\kappa_j) + \Gamma^\star(\sigma_j,\sigma'_j;\kappa_i) - \Gamma^\star(\kappa_i;\kappa_j)\Big] \\ &\qquad - \frac{1}{4}\sum_{i,j} a_{ij}\big(\gamma(\pi_i,\pi_j) - \Gamma^\star(\kappa_i;\kappa_j)\big). \end{aligned}$$

Up to here this was – up to the integration by parts – the same as in the finite case. Now, we have to apply the Dirac measure trick to get the centered factors:

$$
\begin{aligned}
\varphi'(x) = \frac{1}{4}\sum_{ij} a_{ij}\nu_x\Big[\int_{\Sigma^2}\gamma(s,t)\big(\delta_i(ds)-\pi_i(s)p(ds)\big)\big(\delta_j(dt)-\pi_j(t)p(dt)\big)\Big]\\
-\frac{1}{4}\sum_{ij} a_{ij}\nu_x\Big[\int_{\Sigma^4}\Gamma(s,t,s',t')\big(\delta_i^{1,2}(ds,ds')-\kappa_i(s,s')p^{\otimes 2}(ds,ds')\big)\\
\cdot\big(\delta_j^{1,2}(dt,dt')-\kappa_j(t,t')p^{\otimes 2}(dt,dt')\big)\Big]\\
-\frac{1}{4}\sum_{i,j} a_{ij}\big(\gamma(\pi_i,\pi_j)-\Gamma^\star(\kappa_i;\kappa_j)\big)\\
=\frac{1}{4}\sum_i \nu_x\Big[\int_{\Sigma^2}\gamma(s,t)\tilde{\delta}_i(ds)\sum_j a_{ij}\tilde{\delta}_j(dt)\Big]\\
-\frac{1}{4}\sum_i \nu_x\Big[\int_{\Sigma^4}\Gamma(s,t,s',t')\tilde{\delta}_i^{1,2}(ds,ds')\sum_j a_{ij}\tilde{\delta}_j^{1,2}(dt,dt')\Big]\\
-\frac{1}{4}\sum_{i,j} a_{ij}\big(\gamma(\pi_i,\pi_j)-\Gamma^\star(\kappa_i;\kappa_j)\big).
\end{aligned}
$$

Now under assumption 7.4, we calculate as in (7.9):

$$
\begin{aligned}
&\nu_x\Big[\int_{\Sigma^4}\Gamma(s,t,s',t')\tilde{\delta}_N^{1,2}(ds,ds')\sum_i a_{iN}\tilde{\delta}_i^{1,2}(dt,dt')\Big]\\
&\quad=\sum_{k,l} M_{k,l}\nu_x\Big[\int_{\Sigma^2} r_k(s)r_l(s')\tilde{\delta}_N^{1,2}(ds,ds')\\
&\qquad\cdot\int_{\Sigma^2} r_k(t)r_l(t')\sum_j a_{Nj}\tilde{\delta}_j^{1,2}(dt,dt')\Big]\\
&\quad\le\sum_{k,l}|M_{k,l}|\sqrt{\nu_x\Big[\Big(\int_{\Sigma^2} r_k(s)r_l(s')\tilde{\delta}_N^{1,2}(ds,ds')\Big)^2\Big]}\\
&\qquad\cdot\sqrt{\nu_x\Big[\Big(\int_{\Sigma^2} r_k(t)r_l(t')\sum_j a_{Nj}\tilde{\delta}_j^{1,2}(dt,dt')\Big)^2\Big]}\\
&\quad\le 2K\,V(\mathbf{a}_{\bullet N}).
\end{aligned}
$$

The analogue calculation as this also bounds the corresponding term with γ by $2K\,U(\mathbf{a}_{\bullet N})$. Then, this gives:

$$\begin{aligned}|\varphi'(x) + \frac{1}{4}\sum_{i,j} a_{ij}\big(\gamma(\pi_i,\pi_j) - \Gamma^\star(\kappa_i;\kappa_j)\big)| &\leq \tfrac{K}{2}U(\mathbf{a}_{\bullet N}) + \tfrac{K}{2}V(\mathbf{a}_{\bullet N})\\ &\leq L\sum_i w_i.\end{aligned}$$

By using:

$$\varphi(0) = \sum_{i\leq N} \mathbb{E}\log \mathrm{Tr}_s \exp\big(Y_i(s) + \Phi^{(\mathbf{C},i)}(s)\big)$$

we finish the generalization of the proof to the GRF setting. □

8. Lowering the Temperature

To give an outlook of what might happen when β becomes bigger, we perform the standard heuristics for this model. The conjectures we gain are open for further work.

First, the Aizenman-Sims-Starr ansatz will be performed by introducing the so-called Random Overlap Structures. Then, a special instance of ROSt is examined, the Ruelle probability cascades. This leaves us with a conjecture on the Parisi-type formula for the free energy at all temperatures. One inequality can easily be done for some instances of our model.

Afterwards, this brings us to understand up to what β the so-called replica symmetric free energy formula in Theorem 4.3 should hold, that is, to describe the whole high temperature regime. For some examples we evaluate this numerically.

We will do this only for the Hamiltonian as stated in (1.2), and not in the more general version of (4.5).

8.1. Random Overlap Structures and Aizenman-Sims-Starr

The famous Aizenman-Sims-Starr ansatz was given in [2]. It introduced the so-called Random Overlap Structures (ROSt). In our setting, we have to adjust them. That is, a ROSt consists of the following data:

- A State Space A, finite or countable infinite, equipped with an average Tr, that is $\mathrm{Tr}_\alpha 1 = 1$,
- 'Overlaps' $Q_{\alpha\alpha'}$, $\alpha, \alpha' \in A$: random symmetric matrices whose entries add up to 1:

$$Q_{\alpha\alpha'} = \big(Q_{\alpha\alpha'}(s,s')\big)_{s,s'\in\Sigma}, \qquad \sum_{s,s'} Q_{\alpha\alpha'}(s,s') = 1, \qquad \alpha,\alpha' \in A$$

Furthermore, $Q_{\alpha\alpha}$ have to be diagonal matrices for all $\alpha \in A$. We write $q_\alpha(s)$ for $Q_{\alpha\alpha}(s,s)$. Note that the diagonal then sums up to 1.

- Random non-negative weights $\eta_\alpha \geq 0$, s.t. $\sum_{\alpha\in A} \eta_\alpha < \infty$.

Our generalization of the SK model then corresponds to $A = \Sigma^N$ and:

$$\operatorname{Tr}_\alpha f(\alpha) = \sum_\alpha f(\alpha) \prod_i p(\alpha_i), \qquad \eta_\alpha = \exp\left[\tfrac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij}(\alpha_i,\alpha_j)\right]$$
$$Q_{\alpha\alpha'}(s,s') = L_N^{\alpha,\alpha'}(s,s') = \frac{1}{N}|\{\, i \leq N \mid (\alpha_i,\alpha_i') = (s,s')\,\}|.$$

Observe that $Q_{\alpha\alpha}(s,s') = \delta_{ss'} \cdot L_N^\alpha(s)$ is a diagonal matrix. We will call this ROSt $\mathcal{R}_{\mathrm{SK}}$. Another very important example will be the Ruelle probability cascades discussed later.

Given a ROST $\mathcal{R}$, we define the families of centered Gaussians κ^α and $y_i^\alpha(s)$, $\alpha \in A, s \in \Sigma, i \leq M$, s.t.:

$$\mathbb{E}\kappa^\alpha \kappa^{\alpha'} = \sum_{s,t,s',t'} \Gamma(s,t,s',t') Q_{\alpha\alpha'}(s,s') Q_{\alpha\alpha'}(t,t') = \Gamma^\star(Q_{\alpha\alpha'};Q_{\alpha\alpha'}),$$
$$\mathbb{E}y_i^\alpha(s) y_j^{\alpha'}(s') = \sum_{t,t'} \Gamma(s,t,s',t') Q_{\alpha\alpha'}(t,t') \cdot \delta_{ij} = \Gamma^\star(Q_{\alpha\alpha'};s,s') \cdot \delta_{ij}.$$

Both families have to be independent of each other.

Given a ROSt $\mathcal{R}$ we define the smart path Hamiltonian for $\alpha \in A$ and $\tau \in \Sigma^M$:

$$\begin{aligned} H_M(\tau,\alpha,x) &:= H_M(\tau,\alpha,x,\mathcal{R}) \\ &= \tfrac{\sqrt{x}}{\sqrt{M}} \sum_{i<j\leq M} g_{ij}(\tau_i,\tau_j) + \sqrt{\tfrac{xM}{2}}\kappa^\alpha + \sqrt{1-x}\sum_i y_i^\alpha(\tau_i), \\ \hat{G}_M(x,\mathcal{R}) &:= \tfrac{1}{M}\mathbb{E}\left[\log \operatorname{Tr}_{\alpha,\tau} \eta_\alpha e^{\beta H_M(\tau,\alpha,x)} - \log \operatorname{Tr}_\alpha \eta_\alpha e^{\beta\sqrt{\frac{M}{2}}\kappa^\alpha} \right]. \end{aligned}$$

Let $\langle \cdot \rangle_{x,\mathcal{R}}$ and $\nu_x(\cdot) = \nu_{x,\mathcal{R}}(\cdot)$ be defined as usual. Then:

$$\begin{aligned}
\frac{d\hat{G}_M}{dx} &= \tfrac{\beta}{2}\nu_x\Big[\tfrac{1}{\sqrt{xM}}\sum_{i<j\le M} g_{ij}(\tau_i,\tau_j) + \sqrt{\tfrac{M}{2x}}\kappa^\alpha - \tfrac{1}{\sqrt{1-x}}\sum_i y_i^\alpha(\tau_i)\Big] \\
&= \tfrac{\beta^2}{2}\nu_x\Big[\tfrac{1}{M}\sum_{i<j\le M}\{\gamma(\tau_i,\tau_j) - \Gamma(\tau_i,\tau_j,\tau_i',\tau_j')\} + \tfrac{M}{2}\gamma(q_\alpha,q_\alpha) \\
&\qquad - \tfrac{M}{2}\Gamma^\star(Q_{\alpha\alpha'};Q_{\alpha\alpha'}) - \sum_i \gamma(q_\alpha,\tau_i) + \sum_i \Gamma^\star(Q_{\alpha\alpha'};\tau_i,\tau_i')\Big] \\
&= M\tfrac{\beta^2}{4}\nu_x\Big[\sum_{s,t}\gamma(s,t)\big\{L_M^\tau(s)L_M^\tau(t) + q_\alpha(s)q_\alpha(t) - 2q_\alpha(s)L_M^\tau(t)\big\} \\
&\qquad - \sum_{s,t,s',t'}\Gamma(s,t,s',t')\big\{L_M^\tau(s,s')L_M^\tau(t,t') + Q_{\alpha\alpha'}(s,s')Q_{\alpha\alpha'}(t,t') \\
&\qquad - 2Q_{\alpha\alpha'}(s,s')L_M^\tau(t,t')\big\}\Big] + o(M) \\
&= M\tfrac{\beta^2}{4}\nu_x\Big[\gamma(L_M^\tau - q_\alpha, L_M^\tau - q_\alpha) \\
&\qquad - \sum_{s,t,s',t'}\Gamma^\star(L_M^\tau - Q_{\alpha\alpha'}; L_M^\tau - Q_{\alpha\alpha'})\Big] + o(M).
\end{aligned}$$

Again, in our case, this is not decisive in general. But, if this is non-negative, then we have:

$$\tfrac{1}{M}\mathbb{E}\log Z_M = \hat{G}_M(1,\mathcal{R}) \le \inf_{\mathcal{R}} \hat{G}_M(0,\mathcal{R})$$

Therefore, in the same fashion as in Theorem 3.2, this yields:

Theorem 8.1. *Assume Γ is such that the quadratic form $Q_\Gamma(f) \le 0$ for all symmetric $|\Sigma| \times |\Sigma|$-matrices f s.t. $\sum_{s,s'} f(s,s') = 0$ as in (3.2). Then, we have:*

$$\lim_{N\to\infty}\tfrac{1}{N}\mathbb{E}\log Z_N \le \lim_{M\to\infty}\inf_{\mathcal{R}}\hat{G}_M(0,\mathcal{R}),$$

where Z_N is the partition function of the multivariate SK model with covariance matrix Γ and the infimum is over all ROSts for Γ.

Actually, the other inequality is not very difficult to prove, since in the situation of the theorem we have already proved super-additivity. The details can for instance be seen in [3, Theorem 4.5].

It seems that for models with $\Gamma^\star$ being positive definite, this should also hold based on the methods of Theorem 3.3. At least this inequality has been performed for the d-component spin SK model in [8] and [4] which took quite some effort. Anyway, in general, there is no reason to doubt:

Conjecture 8.2. *For all* (Σ, Γ, p) *and for all* $\beta \geq 0$.

$$\lim_{N\to\infty} \tfrac{1}{N}\mathbb{E}\log Z_N = \lim_{M\to\infty} \inf_{\mathcal{R}} \hat{G}_M(0,\mathcal{R})$$

This is not a big help yet, because the infimum might be even harder to evaluate in general. But the Ruelle probability cascades ROSt is thought of giving a calculable infimum. For the spherical SK model, that has been proved.

Remarkably, for models with $Q_\Gamma(f) = 0$ for all corresponding f, we hereby see that all differences between different ROSts are vanishing as $N \to \infty$. We already saw that the Toy Model 2.3.3 has this property. But there are many others including:

$$\Gamma = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8.1}$$

which gives:

$$Q_\Gamma \begin{pmatrix} a & c \\ c & b \end{pmatrix} = 4c\,(a + 2b + c) = 0$$

for all the f we are considering.

8.2. Ruelle Probability Cascade

A very important example of ROSt's are the Ruelle probability cascades. Actually, for the standard and the spherical SK model, they are proved to give the infimum, and hence proving the Parisi formula.

In our setting, they are to be defined as follows. First, let $K \in \mathbb{N}$ and $0 = Q_0 \overset{\Gamma}{\precsim} Q_1 \overset{\Gamma}{\precsim} \cdots \overset{\Gamma}{\precsim} Q_K \overset{\Gamma}{\precsim} Q_{K+1}$ be symmetric non-negative matrices with entries summing up to one, Q_{K+1} diagonal with trace 1, and $0 = m_0 < m_1 < \cdots < m_{K-1} < m_K = 1$.

Then, the Ruelle probability cascade ROSt $\mathcal{R}_{\text{Ruelle}}(K, \mathbf{Q}, \mathbf{m})$ is given by $A = \mathbb{N}^K$ and $Q_{\alpha\alpha'} = Q_{d(\alpha,\alpha')}$ where $d(\alpha, \alpha') := \min\{i \mid \alpha_i \neq \alpha'_i\}$. For the weights, we define:

$$\eta_\alpha = \eta^1_{\alpha_1} \eta^2_{\alpha_1,\alpha_2} \cdots \eta^K_{\alpha_1,\ldots,\alpha_K}$$

where the $\eta^i_{\alpha_1,\ldots,\alpha_i}$ are the points of a $\text{PPP}(m_i t^{-m_i-1})$ in decreasing ordering – the point processes being independent for all the different indexes.

Now, the famous Parisi formula is given by

$$\inf_{K,\mathbf{Q},\mathbf{m}} \hat{G}_M\big(0, \mathcal{R}_{\text{Ruelle}}(K, \mathbf{Q}, \mathbf{m})\big)$$

and we evaluate this. First, we let

$$Y_{K+1} := \operatorname{Tr}_s e^{\beta \sum_{i=0}^K g_i(s)},$$

where $(g_i)_{i=0}^K$ are i.i.d. Gaussian fields with covariance matrices $\Gamma_{Q_{i+1}-Q_i}$. Then, we define recursively:

$$Y_i := \big[\mathbb{E}_{g_i}(Y_{i+1}^{m_i})\big]^{1/m_i},$$

where $\mathbb{E}_{g_i}$ is the integration of just g_i. Remark that Y_i only depends on the variables $g_0, \ldots, g_{i-1}$.

Lemma 8.3. *Let $\mathcal{R}_{Ruelle}(K, \mathbf{Q}, \mathbf{m})$ be a Ruelle probability cascade. Then:*

$$\hat{G}_M\big(0, \mathcal{R}_{Ruelle}(K, \mathbf{Q}, \mathbf{m})\big) = \mathbb{E} \log Y_1 - \frac{\beta^2}{4} \sum_{i=1}^K m_i \big(\Gamma^\star(Q_{i+1}; Q_{i+1}) - \Gamma^\star(Q_i; Q_i)\big),$$

Proof. First, recall the distributional equation

$$\{y_i \cdot \eta_i\}_i \overset{L}{=} \{\big(\mathbb{E}(y^m)\big)^{1/m} \cdot \eta_i\}_i,$$

if $\{\eta_i\}_i$ is a $\text{PPP}(mt^{-m-1})$ and y_i are i.i.d. random variables with finite m-th moment, independent of the η_i.

Next, we will denote by $\alpha^k := (\alpha_1, \ldots, \alpha_k)$ the restriction to the first k entries. Hence, $\alpha^1 = \alpha_1$ and $\alpha^K = \alpha$.

We have to calculate:

$$\hat{G}_M\big(0, \mathcal{R}_{\text{Ruelle}}(K, \mathbf{Q}, \mathbf{m})\big) = \tfrac{1}{M}\mathbb{E}\log \operatorname{Tr}_{\alpha\in A}\eta_\alpha \operatorname{Tr}_s e^{\beta y^\alpha(s)} - \tfrac{1}{M}\mathbb{E}\log \operatorname{Tr}_\alpha \eta_\alpha e^{\beta\sqrt{\frac{M}{2}}\kappa^\alpha}. \tag{8.2}$$

We first handle the second summand. Let $\kappa^\alpha := \sum_{k=1}^K z_k(\alpha^k)$ with $z_k(\alpha^k)$ is of variance $\beta^2\frac{M}{2}\big(\Gamma^\star(Q_k;Q_k) - \Gamma^\star(Q_{k-1};Q_{k-1})\big)$, independently for all $\alpha^k \in \mathbb{N}^k$. Then:

$$\eta_\alpha e^{\beta\sqrt{\frac{M}{2}}\kappa^\alpha} = \eta_{\alpha^1}e^{z_1(\alpha^1)}\cdot\eta_{\alpha^2}e^{z_2(\alpha^2)}\cdots\eta_{\alpha^K}e^{z_K(\alpha^K)}.$$

Now, for each factor $\eta_{\alpha^k}e^{z_k(\alpha^k)}$, this has as stated above the same law as:

$$\eta_{\alpha^k}\left[\mathbb{E}e^{m_k z_k(\alpha^k)}\right]^{1/m_k} = \eta_{\alpha^k}e^{\beta^2\cdot\frac{m_k M}{4}\left(\Gamma^\star(Q_k;Q_k)-\Gamma^\star(Q_{k-1};Q_{k-1})\right)}$$

We do this sequentially for every factor, collect terms, and obtain:

$$\eta_\alpha\cdot e^{\beta\sqrt{\frac{M}{2}}\kappa^\alpha} = \eta_\alpha\cdot e^{\beta^2\cdot\frac{M}{4}\sum_{k=1}^K m_k\left(\Gamma^\star(Q_k;Q_k)-\Gamma^\star(Q_{k-1};Q_{k-1})\right)}$$

Observe that the second factor is deterministic, and hence explains the second term in the Parisi formula.

Now, for the first term in (8.2), because of the definition of the Ruelle probability cascade ROSt, we have:

$$y^\alpha(s) = \sum_{k=0}^K g^{(k)}_{\alpha^k}(s),$$

where $g^{(k)}_{\alpha^k}$ is a Gaussian field with covariance matrix $\Gamma_{Q_{k+1}-Q_k}$, and being independent for each $\alpha^k \in \mathbb{N}^k$. Hence we have, conditioned on $\eta_{\alpha^1},\ldots,\eta_{\alpha^{K-1}}$ and the g's up to level $K-1$:

$$\begin{aligned}\{\,\eta_\alpha \operatorname{Tr}_s e^{\beta\sum_{k=0}^K g^{(k)}_{\alpha^k}(s)}\,\}_{\alpha_K} &= \{\,\eta_\alpha \operatorname{Tr}_s e^{\beta\sum_{k=0}^{K-1} g^{(k)}_{\alpha^k}(s)}\cdot e^{\beta g^{(K)}_{\alpha^K}(s)}\,\}_{\alpha_K}\\ &\overset{L}{=} \{\,\eta_\alpha\cdot X_{K-1}\,\}_{\alpha_K},\end{aligned}$$

where

$$X_{K-1} := C_K\big(\beta \sum_{i=0}^{K-1} g_\alpha^{(i)}(s)\big), \quad C_K\big(\xi(\cdot)\big) := \Big[\mathbb{E}_{g_i}\big(\operatorname{Tr}_s e^{\xi(s)+\beta g_i(s)}\big)^{m_i}\Big]^{1/m_i}.$$

Hence, X_{K-1} is of course the m_i-th moment of the random variable

$$\operatorname{Tr}_s e^{\beta \sum_{i=0}^K g_\alpha^{(i)}(s)}$$

conditioned on the g's up to level $K-1$. On the other hand, $X_{K-1} \stackrel{L}{=} Y_{K-1}$ is a random variable depending only on the g's at levels less than K. Therefore, using the same recursively, one arrives at:

$$\eta_\alpha \operatorname{Tr}_s e^{\beta \sum_{k=0}^K g_{\alpha k}^{(k)}(s)} \stackrel{L}{=} X_1 \eta_\alpha,$$

with $X_1 \stackrel{L}{=} Y_1$. Hence, this leads to the first summand in the Parisi formula. In summery we have:

$$\begin{aligned}\hat{G}_M\big(0, \mathcal{R}_{\text{Ruelle}}(K, \mathbf{Q}, \mathbf{m})\big) = \mathbb{E}\log Y_1 + \tfrac{1}{M}\mathbb{E}\log \frac{\operatorname{Tr}_\alpha \eta_\alpha}{\operatorname{Tr}_\alpha \eta_\alpha} \\ - \frac{\beta^2}{4}\sum_{i=1}^K m_i\big(\Gamma^\star(Q_{i+1};Q_{i+1}) - \Gamma^\star(Q_i;Q_i)\big),\end{aligned}$$

Now, the second summand is only vanishing if $\operatorname{Tr}_\alpha \eta_\alpha$ is finite, which is only given for $m_K < 1$, contrary to our assumptions. Therefore, we first perform the calculation for $m_K < 1$, and then let $m_K \to 1$. $\square$

This motivates:

Conjecture 8.4 (Parisi Formula)**.**

$$\begin{aligned}&\lim_{N\to\infty} \tfrac{1}{N} \log Z_N(\beta, \Gamma) \\ &= \inf_{K,\mathbf{Q},\mathbf{m}} \Big[\mathbb{E}\log Y_1 - \frac{\beta^2}{4}\sum_{i=1}^K m_i\big(\Gamma^\star(Q_{i+1};Q_{i+1}) - \Gamma^\star(Q_i;Q_i)\big)\Big]. \qquad (8.3)\end{aligned}$$

One half of this is given for some models in Theorem 8.1. The most difficult part of this conjecture of course is to obtain that the infimum over all ROSts is already given by the infimum over the Ruelle probability cascade ROSts. This has not even been proved for the d-component spin SK model.

8.3. High-Temperature Region

Now that we have some insight on the form of the Parisi formula in our case, we can get a feeling of what might be the **High Temperature Region**. We are following here the ideas developed in [20], [11], and [17, p. 13.3]. There, the notion of High Temperature Region is defined as those β where the **replica symmetric free energy formula** (given by the formula in Theorem 4.3) is the minimizer of (8.3).

To calculate this, we first define the Parisi functional:

$$\mathcal{P}_K(\mathbf{Q},\mathbf{m}) := \mathbb{E}\log Y_1 - \frac{\beta^2}{4}\sum_{i=1}^{K} m_i\big((Q_{i+1};Q_{i+1}) - \Gamma^\star(Q_i;Q_i)\big),$$
$$\mathcal{P}_K(\Gamma) := \inf_{\mathbf{Q},\mathbf{m}} \mathcal{P}_K(\mathbf{Q},\mathbf{m}), \qquad \mathcal{P}(\Gamma) := \inf_K \mathcal{P}_K(\Gamma)$$

Note that the formula in Theorem 4.3 is

$$\mathrm{RS}(\Gamma) := \mathcal{P}_1(\mathbf{Q},\mathbf{m}) \qquad \text{with} \qquad \mathbf{Q} = \big(Q,P\big) \qquad \text{and} \qquad \mathbf{m} = (0,1),$$

where

$$Q := \kappa \qquad \text{and} \qquad P := \mathrm{diag}(\pi)$$

are the order parameters with new denotations in this setting. This implies $\mathcal{P}_1(\Gamma) \le RS(\Gamma)$. In the case of Theorem 8.1, we know that the free energy is bound by $\mathcal{P}_1(\Gamma)$. This implies that the order parameters are minimizing the inf in $\mathcal{P}_K(\Gamma)$ for β small enough. In view of the conjectures above we henceforth assume that this is true for all Γ.

Now, following the idea of Panchenko, we define the **high temperature region** as the set of β at which the free energy equals $\mathcal{P}_1$, that is $\inf_K \mathcal{P}_K(\Gamma)$ is already achieved for $K = 1$. On the other hand, the high temperature region stops as soon as $\mathcal{P}_2(\Gamma) < \mathcal{P}_1(\Gamma) = \mathrm{RS}(\Gamma)$. The idea is to see for what parameters this happens.

Therefore, we consider the following second level replica symmetry breaking:

$$K = 2, \qquad \mathbf{Q} = (0,Q,A,P), \qquad \mathbf{m} = (0,m,1).$$

Thus, the parameters new on this level are only A and m. Consider

$$\Psi(m,A) := \mathcal{P}_2(\mathbf{Q},\mathbf{m}) = -\tfrac{\beta^2}{4}\big[\Gamma^\star(P;P) - \Gamma^\star(A;A) + m\Gamma^\star(A;A) \\ - m\Gamma^\star(Q;Q)\big] + \tfrac{1}{m}\mathbb{E}\log\mathbb{E}_1\big[\,\mathrm{Tr}_s\, e^{Y'(s)}\big]^m,$$

where

$$Y'(s) := g_0(s) + g_1(s) + \Phi_{P-A}(s),$$
$$\Phi_{P-A}(s) = \tfrac{1}{2}\big(\Gamma^\star(s,s;P) - \Gamma^\star(s,s;A)\big).$$

Now, trivially $\Psi(1,A) = \mathrm{RS}(\Gamma)$ does not depend on A. In the high temperature region $\Psi(m,A)$ should attain its minimum at $(1,Q)$. Therefore, the idea is to look at the following quantity for all A:

$$\begin{aligned} f(A) := \tfrac{\partial\Psi(m,A)}{\partial m}\Big|_{m=1} &= -\tfrac{\beta^2}{4}\Big(\Gamma^\star(A;A) - \Gamma(Q,Q)\Big) - \mathbb{E}\log\mathbb{E}_1\big[\operatorname{Tr}_s e^{Y'(s)}\big] \\ &+ \mathbb{E}\frac{\mathbb{E}_1\big[\log\big(\big[\operatorname{Tr}_s e^{Y'(s)}\big]\big)\cdot \operatorname{Tr}_s e^{Y'(s)}\big]}{\mathbb{E}_1\big[\operatorname{Tr}_s e^{Y'(s)}\big]} \\ &= -\tfrac{\beta^2}{4}\Big(\Gamma^\star(A;A) - \Gamma(Q,Q)\Big) + \mathbb{E}\tfrac{\operatorname{Tr}_s e^{Y'(s)}}{\mathbb{E}_1 \operatorname{Tr}_s e^{Y'(s)}}\cdot\log\big[\tfrac{\operatorname{Tr}_s e^{Y'(s)}}{\mathbb{E}_1 \operatorname{Tr}_s e^{Y'(s)}}\big] \end{aligned}$$

Following [11], we see that necessary condition for $\mathcal{P}(\Gamma) = \mathcal{P}_1(\Gamma)$ is:

$$f(A) \le 0 \qquad \text{for all} \qquad Q \overset{\Gamma}{\precsim} A \overset{\Gamma}{\precsim} P, \tag{8.4}$$

since $m = 1$ should be a minimum for all A.

The famous AT-Line corresponds to the Hessian of f being negative semidefinite at $A = Q$. Refer to [11] for the fact that those two conditions are not equivalent in some versions of the spherical SK model.

Now, we investigate this bound for two examples.

8.3.1. Example: Field of Independent Gaussians

Here, $\Sigma = \{\pm 1\}$ and $\Gamma(s,t,s',t') = 1_{(s,t)=(s',t')}$ as we saw in Section 2.3.1. There, we established $\pi(+1) = \pi(-1) = \frac{1}{2}$ and $Q = \kappa = \begin{pmatrix} q & q' \\ q' & q \end{pmatrix}$ with $q' = \frac{1}{2} - q$ and q satisfying (2.2). We already know by our proof that this model exhibits high temperature behaviour for $\beta < \frac{1}{\sqrt{8K_\Gamma}} = \frac{1}{8}$. To obtain some understanding on how far away from the correct β_c this is, we calculate the above conditions.

The first thing to do is to evaluate Γ_A:

$$\Gamma_A(s,s') = \begin{pmatrix} \operatorname{Tr} A & 0 \\ 0 & \operatorname{Tr} A \end{pmatrix}$$

Now, let A and B be two matrices with entries summing up to 1 and denote their traces by a, b resp. Then, we have:

$$\left(\Gamma^\star(s,s';B) - \Gamma^\star(s,s';A)\right)_{s,s'} = (b-a) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is positive semidefinite for $\operatorname{Tr} B = b \geq a = \operatorname{Tr} a$.

Now, $g_i(+1)$ and $g_i(-1)$ are independent Gaussians with variances $\operatorname{Tr} Q_{i+1} - \operatorname{Tr} Q_i$. Furthermore,

$$\Gamma^\star(A;A) = a^2 = (\operatorname{Tr} A)^2.$$

Of course $\operatorname{Tr} P = 1$ and $\operatorname{Tr} Q = 2q$. Now, for any such matrix A with $a = \operatorname{Tr} A \in [2q, 1]$:

$$\begin{aligned}
Y'(s) &= \beta g_0(s) + \beta g_1(s) + \tfrac{\beta^2}{2}(1-a), \qquad Y''(s) = \beta g_0(s) + \beta g_1(s) \\
Z &:= \operatorname{Tr}_s e^{Y'(s)} = \operatorname{Tr}_s e^{\beta g_0(s) + \beta g_1(s) + \beta^2 \Phi_{P-A}(s)} = e^{\frac{\beta^2}{2}(1-a)} \cdot \operatorname{Tr}_s e^{Y''(s)} \\
X &:= \mathbb{E}_1 \operatorname{Tr}_s e^{Y'(s)} = \operatorname{Tr}_s e^{\beta g_0(s) + \beta^2 \Phi_{P-Q}(s)} = e^{\beta^2(\frac{1}{2}-q)} \cdot \operatorname{Tr}_s e^{\beta g_0(s)} \\
\frac{Z}{X} &= e^{\frac{\beta^2}{2}(2q-a)} \cdot \frac{\operatorname{Tr}_s e^{Y''(s)}}{\operatorname{Tr}_s e^{\beta g_0(s)}}, \qquad \mathbb{E}\frac{Z}{X} = \mathbb{E}\mathbb{E}_1 \frac{Z}{X} = 1 \\
f(A) &= -\tfrac{\beta^2}{4}[a^2 - 4q^2] + \mathbb{E}\tfrac{Z}{X} \cdot \log(\tfrac{Z}{X})
\end{aligned}$$

Note that this only depends on the trace a of A. Numerical evaluations of this quantity suggest that the replica symmetric formula only holds up to $\beta \approx 0.75$ as then $f(A)$ becomes positive for some $\operatorname{Tr} a > 2q$, c.f. Figure 8.3.1.

8.3.2. Example: Volunteer Model

This is the model where $\Gamma(1,1,1,1) = 1$ and all other values are zero. Again, $\pi(+1) = \pi(-1) = \frac{1}{2}$ and $Q = \kappa = \begin{pmatrix} q & q' \\ q' & q \end{pmatrix}$ with $q' = \frac{1}{2} - q$ and q. Now, we first evaluate:

$$\Gamma_A = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $0 \overset{\Gamma}{\preceq} A - B$ iff $a_{11} \geq b_{11}$, and we have

$$\Gamma^\star(A;A) = a_{11}^2.$$

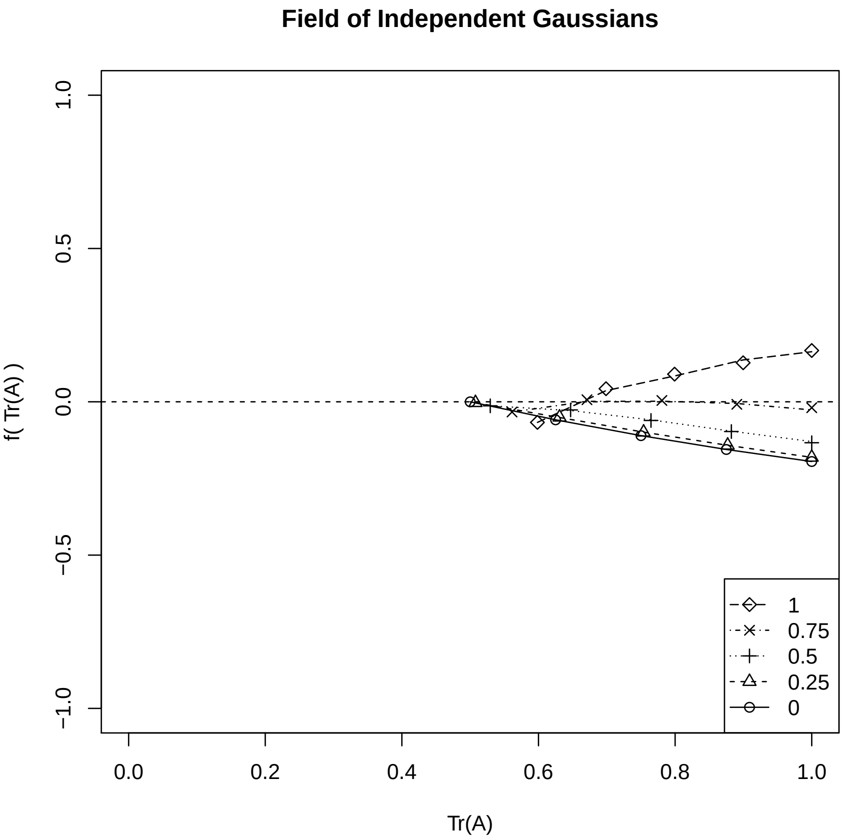

Figure 8.1.: $f(a)$ for the field of independent Gaussians model.

Now, let z_0 and z_1 be independent standard Gaussians. Then, to look at the level one replica symmetry breaking formula, we set for any A with $a = a_{11} \in [q, 1]$:

$$\begin{aligned}
Y'(s) &= \beta g_0(s) + \beta g_1(s) + \tfrac{\beta^2}{2}(1-a) \\
&= \beta(\sqrt{a-q}z_0 + \sqrt{1-a}z_1)\, 1_{s=1} + \tfrac{\beta^2}{2}(1-a), \\
Y''(s) &= \beta g_0(s) + \beta g_1(s) = \beta(\sqrt{a-q}z_0 + \sqrt{1-a}z_1)\, 1_{s=1} \\
Z :&= \mathrm{Tr}_s\, e^{Y'(s)} = \mathrm{Tr}_s\, e^{\beta g_0(s) + \beta g_1(s) + \beta^2 \Phi_{P-A}(s)} \\
&= e^{\frac{\beta^2}{2}(1-a)} \cdot \frac{e^{\beta(\sqrt{a-q}z_0 + \sqrt{1-a}z_1)} + 1}{2} \\
X :&= \mathbb{E}_1\, \mathrm{Tr}_s\, e^{Y'(s)} = \mathrm{Tr}_s\, e^{\beta g_0(s) + \beta^2 \Phi_{P-Q}(s)} = e^{\frac{\beta^2}{2}(1-q)} \cdot \frac{e^{\beta\sqrt{a-q}z_0} + 1}{2} \\
\frac{Z}{X} &= e^{\frac{\beta^2}{2}(q-a)} \cdot \frac{\mathrm{Tr}_s\, e^{Y''(s)}}{\mathrm{Tr}_s\, e^{\beta g_0(s)}}, \qquad \mathbb{E}\frac{Z}{X} = \mathbb{E}\mathbb{E}_1 \frac{Z}{X} = 1 \\
f(A) &= -\tfrac{\beta^2}{4}[a^2 - q^2] + \mathbb{E}\tfrac{Z}{X} \cdot \log(\tfrac{Z}{X})
\end{aligned}$$

The numerical evaluation of f here suggests that the critical β is approximately $\beta_c = 1$, c.f. Figure 8.3.2, whereas we rigorously know it to be valid for $\beta < \beta_0 = \frac{1}{2\sqrt{2}}$.

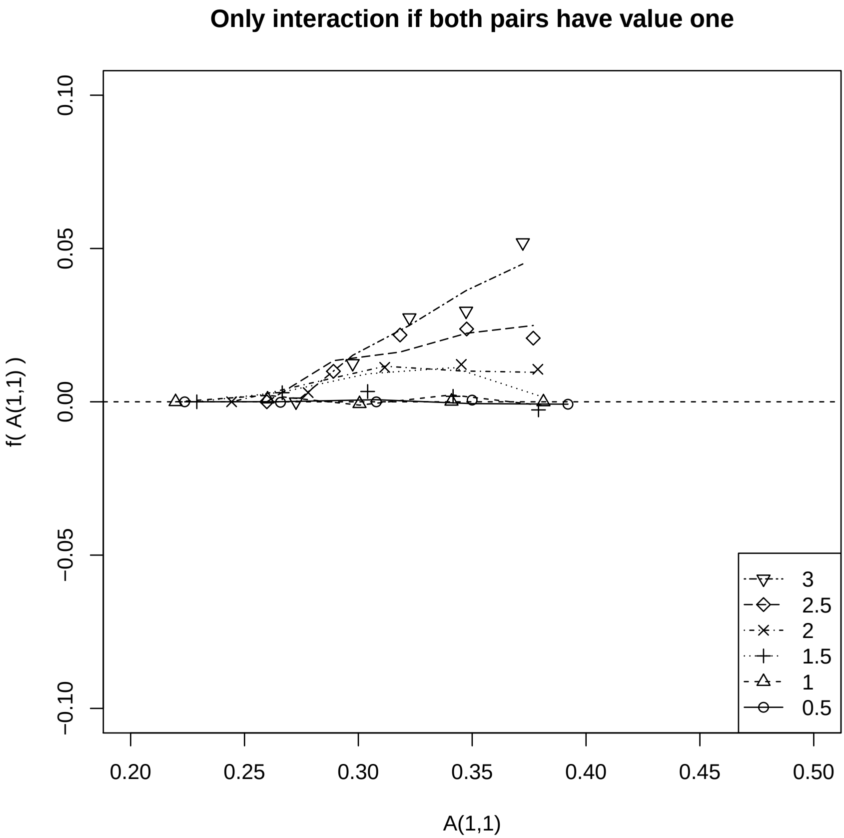

Figure 8.2.: $f(a)$ for the volunteer model.

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Notation

Notation	Meaning	Page
γ	The variance corresponding to the covariance matrix Γ, i.e. the diagonal of Γ	
$\Gamma^\star(s,s';t,t')$	$\Gamma(s,t,s',t')$	10
Γ_A	$\big(\Gamma^\star(s,s';A)\big)_{s,s'}$	10
$0 \overset{\Gamma}{\preccurlyeq} A$	Γ_A is positive semidefinite	10
κ	The solution of the fixed point equation: $\kappa(s,s') = \mathbb{E}\Pi(s)\Pi(s')$	13
π	The solution of the fixed point equation $\pi(s) = \mathbb{E}\Pi(s)$, that is $\pi(s) = \sum_{s'} \kappa(s,s')$.	
$\langle\, f \,\rangle$	The quenched average of the function f	10
$\nu(f)$	$\mathbb{E}\langle\, f \,\rangle$: The averaged expectation of the function f	10
$\delta_i^\ell(s)$	$1_{\sigma_i^\ell = s}$	11
$L_N^\ell(s)$	$\frac{1}{N}\sum_{i=1}^N \delta_i^\ell(s)$	11
$L_N^{\ell,\ell'}(s,s')$	$\frac{1}{N}\sum_{i=1}^N \delta_i^\ell(s)\delta_i^{\ell'}(s)$	11
$\mu_i(s)$	$\langle\, \delta_i(s) \,\rangle$	11
$\Gamma_{\boldsymbol{\kappa}}^{(\mathbf{A},j)}(s,s')$	$\sum_j a_{ij}\Gamma_{\kappa_j}(t,t')$	56
$\Phi^{(i)}(s)$	$\Phi_{\kappa_i}(s)$	74
$\Phi_{\boldsymbol{\kappa}}^{(\mathbf{A},i)}(s)$	$\sum_j a_{ij}\Phi_{\kappa_j}(s)$	56
$\mathrm{ph}_{t\,/\,\Sigma}\big(F(t)\big)$	$\frac{p(t)\cdot e^{F(t)}}{\sum_{t'\in\Sigma} p(t')\cdot e^{F(t')}}, \qquad \forall t \in \Sigma$	92
$O(\boldsymbol{\alpha})$	Big O of the right hand side of Theorem 4.4.	93
$\mathbf{a}_{\bullet i}$	The i-th row of the matrix $\mathbf{A}$.	74
K, L	A constant depending on Γ and p alone	17